Invariant Lie polynomials in two and three variables

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Abstract

In 1949, Wever observed that the degree d of an invariant Lie polynomial must be a multiple of the number q of generators of the free Lie algebra. He also found that there are no invariant Lie polynomials in the following cases: $q = 2$, $d = 4$; $q = 3$, $d = 6$; $d = q \geq 3$. Wever gave a formula for the number of invariants for $q = 2$ in the natural representation of $sl(2)$. In 1958, Burrow extended Wever's formula to $q > 1$ and $d = mq$ where $m > 1$.

In the present thesis, we concentrate on finding invariant Lie polynomials (simply called Lie invariants) in the natural representations of $sl(2)$ and $sl(3)$, and in the adjoint representation of $sl(2)$.

We first review the method to construct the Hall basis of the free Lie algebra and the way to transform arbitrary Lie words into linear combinations of Hall words. To find the Lie invariants, we need to find the nullspace of an integer matrix, and for this we use the Hermite normal form. After that, we review the generalized Witt dimension formula which can be used to compute the number of primitive Lie invariants of a given degree.

Secondly, we recall the result of Bremner on Lie invariants of degree ≤ 10 in the natural representation of $sl(2)$. We extend these results to compute the Lie invariants of degree 12 and 14. This is the first original contribution in the present thesis.

Thirdly, we compute the Lie invariants in the adjoint representation of $sl(2)$ up to degree 8. This is the second original contribution in the present thesis.

Fourthly, we consider the natural representation of $sl(3)$. This is a 3-dimensional natural representation of an 8-dimensional Lie algebra. Due to the huge number of Hall words in each degree and the limitation of computer hardware, we compute the Lie invariants only up to degree 12.

Finally, we discuss possible directions for extending the results. Because there are infinitely many different simple finite dimensional Lie algebras and each of them has infinitely many distinct irreducible representations, it is an open-ended problem.

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To my parents and my wife

Contents

Chapter 1

Introduction

Generally, the basic problem of invariant theory can be described as follows: Let V be a vector space over a field K on which a group G acts linearly. In the polynomial algebra $K[V]$, we consider the subalgebra $K[V]^G$ consisting of all polynomials on V which are invariant under the action of the group G.

1.1 Comments on the references

The following four are the major references for this thesis.

1. The standard reference for Lie algebra is James Humphreys, *Introduction to Lie algebras and representation theory* [10].

2. The standard reference for free Lie algebras is Christophe Reutenauer, *Free Lie algebras* [24].

3. The standard reference for Hall basis is Marshall Hall, *A bases for Free Lie rings and higher commutators in Free groups* [11].

4. All original contributions in the thesis are extension of Murray Bremner, *Lie invariants of degree ten* [3].

Reutenauer also defines the Hall basis in his book [24]. But we refer Hall's definition [11] as the standard one in the thesis which is different from Reutenauer's definition. Besides, there are a few useful references for the thesis which are:

1. References for classical invariant theory are [18], [23], [22].

2. References for representation theory, Lie algebras and free Lie algebras are [9], and [2].

1.2 Classical Invariant Theory

We recall a classical example in invariant theory. (See [22], (pp1-2).)

In the eighteenth century, the Italian mathematician Joseph-Louis Lagrange observed that the quadratic polynomial

$$P(x,y) = ax^2 + 2bxy + cy^2$$

where a, b and c are complex coefficients and x, y are indeterminates, has an interesting property. If we replace x by $x+y$ and y is unchanged, we get

$$\begin{aligned} P(x+y,y) &= a(x+y)^2 + 2b(x+y)y + cy^2 \\ &= ax^2 + 2(a+b)xy + (a+2b+c)y^2 \\ &= Ax^2 + 2Bxy + Cy^2 \end{aligned}$$

where $A = a, B = a+b$ and $C = a+2b+c$. The interesting part is

$$\begin{aligned} AC - B^2 &= a(a+2b+c) - (a+b)^2 \\ &= a^2 + 2ab + ac - a^2 - 2ab - b^2 = ac - b^2. \end{aligned}$$

So $AC - B^2 = ac - b^2$. In fact, it is not hard to find out that we can let $G_k(x) = x + ky$ and $G_k(y) = y$ where k is a complex number. We define $I(P) = ac - b^2$ and obtain the equality

$$I(P) = I(G_k(P))$$

where $G_k(P)(x,y) = P(x+ky,y)$.

A polynomial $P(x,y) = ax^2 + 2bxy + cy^2$ is determined by its coefficients a, b and c. Thus we can represent the polynomials $P(x,y)$ as the elements of a 3-dimensional complex vector space by identifying $ax^2 + 2bxy + cy^2$ with $(a,b,c)^T \in \mathbb{C}^3$. Then $G_k : \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ where $G_k : (a,b,c)^T \longrightarrow (a, ka+b, k^2a + 2kb + c)^T$. If we consider the matrix representation of the linear transformation, we can get the following lower triangular matrix

$$G_k = \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ k^2 & 2k & 1 \end{bmatrix}.$$

The matrices in such a form are closed under multiplication and G_{-k} is the inverse of G_k. The collection of all G_k, where k is some complex number, is a subgroup of the 3×3 general linear group. So $I(P)$ is a polynomial of degree 2 determined by the coefficients and is **invariant** under the group action. This example gives us the original thought about a polynomial which is invariant under the action of a matrix group.

In the nineteenth century, the invariant theory was refereed as "the bridge between algebra and geometry". Mathematicians Hilbert, Boole, Cayley and others made outstanding contributions in this regard. Invariant Theory has been studied ever since. Nowadays, Invariant Theory has many application to problems in Physics and Engineering. It is a blend of many different fields of mathematics as well. For example, representation theory of semisimple (§2.1) Lie groups has its roots in invariant theory. The great mathematician Hilbert's first work was about invariant theory and he proved his famous Basis Theorem [14] in 1888. His research about the finite generation of the algebra of invariants resulted in the creation of a new mathematical discipline, abstract algebra which has become one of the most important fields in pure mathematics nowadays. However, in the present thesis, we focus on the nonassociative algebra of invariant polynomials, which is rarely studied in invariant theory.

1.3 Nonassociative Algebras of Invariants

The classical invariant theory of polynomials is defined in terms of group actions on ordinary polynomial algebras which are also free commutative associative algebras (See [18], [23], [22]). If we remove the commutative assumption, we obtain the free (noncommutative) associative algebras. For example, in §2.2, we will introduce the free associative K-algebra $K\langle A\rangle$ consisting of noncommutative polynomials which are linear combinations over K of words on an alphabet A. Moreover, there exists a smallest submodule L of $K\langle A\rangle$ containing A, where L is closed under the Lie bracket. We can show that this submodule L is a free Lie algebra, which is a nonassociative algebra as well. Therefore invariant polynomials in the free Lie algebra under group actions are nonassociative.

In the present thesis, we want to determine the nonassociative invariants in the representations L of $sl(2)$ and $sl(3)$ where L is a free Lie algebra generated by 2 and 3 elements respectively. Those nonassociative invariants are called the invariant Lie polynomials (simply, Lie invariants). As introduced in Burrow [7], the theory of Lie invariants was first developed by Wever [26], [27]. The importance of Lie invariants for group theory was also exhibited by Wever. It is due to the relationship between a free Lie algebra and the lower series of the free group [20]. Moreover, the rapid growth of the number of Lie invariants of degree mr as a function of m, where r is the number of the free generators (variables), suggests that the ring of the Lie invariants in r variables can probably not be generated by a finite number of invariants. The fact is that the nonassociative algebra K of Lie invariants has infinite dimension since the generalized Witt formula (see §2.5) implies the number of Lie invariants goes to infinity as the degree approaches infinity. So it is most likely impossible to compute all of the Lie invariants in the natural representation of $sl(2)$ or $sl(3)$. Due to the limitation of computer hardware, we compute Lie invariants of reasonable degrees in the thesis. Generally, in the history of the study in invariant theory, nonassociative invariants have not been studied much at all. Most of the results after Chapter 2 are original contributions in the present thesis.

1.4 Main Problem

We begin this thesis by reviewing the results of Bremner's paper *Lie invariants of degree 10*; see reference [3]. This paper determined the Lie invariants of degree less than or equal to 10 in the natural representation of the simple 3-dimensional Lie algebra $sl(2)$; in this case, L is a free Lie algebra generated by 2 elements, say a and b. We extend his work to compute the Lie invariants of degree up to 14 under the natural actions by $sl(2)$.

We next focus on computing the Lie invariants in a free Lie algebra L generated by 3 elements. There are 2 ways to approach this aim. First, we consider a bigger representation of $sl(2)$. We compute the Lie invariants in the adjoint representation of $sl(2)$. To be precise, those Lie polynomials are invariant when the 3 dimensional simple Lie algebra $sl(2)$ acts on a vector space generated by 3 elements. Secondly, we consider the representation of simple (§2.1) finite Lie algebra of higher rank. To be precise, we compute the Lie invariants in 3-dimensional natural representation of the 8-dimensional Lie algebra $sl(3)$ in 3 generators. Please note that the Lie invariants of the same degree in above two cases are totally different.

In all cases, since there are infinitely many Lie invariants as the degree approaches infinity, it is likely to be an open-ended problem.

In Chapter 2, we begin to review some terminologies and give a brief overview of some classical results which are related to the present thesis.

Chapter 2

Some Classical Results

In this thesis, we will approach the problem of Lie invariants from the perspective of the free Lie algebra, or the free Lie ring if the coefficients come from the ring of integers instead of a field. A Lie invariant is a linear combination of the Hall basis or Standard monomials (see [11]), which form a basis of the free Lie algebra. In this chapter, we will give a brief review of some basic definitions and classical results related to the problem considered in this thesis.

2.1 Lie Algebras and Representations

In this section, the basic results of Lie algebras are from the book of Humphreys (see [10]).

Definition 2.1.1. A vector space L over a field F together with a binary operation $L \times L \longrightarrow L$, denoted by $(x, y) \longmapsto [x, y]$, is a **Lie algebra** over F if the following axioms are satisfied:

(1) $[ax + by, z] = a[x, z] + b[y, z]$ and $[z, ax + by] = a[z, x] + b[z, y]$ for all $x, y, z \in L$ and $a, b \in F$, which is also called bilinearity.

(2) $[x, x] = 0$ for all x in L.

(3) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$, which is the famous **Jacobi identity**.

Remark 2.1.1. The Jacobi identity can be rewritten as $[[x, y], z] = [[x, z], y] + [x, [y, z]]$. We will use this equation intensively in Chapter 3 in order to transform any word in a free magma to a Hall word. For more information on free magmas and Hall words, please see §2.4.

Definition 2.1.2. A **subalgebra** of a Lie algebra L is a subspace K of L such that $[x, y] \in K$ for all $x, y \in K$.

Definition 2.1.3. For any associative algebra A, one can construct a Lie algebra $L(A)$. (As a vector space, A is the same as $L(A)$.) The **Lie bracket** of two elements $x, y \in L(A)$ is defined to be $[x, y] = xy - yx$.

We should notice that $[x+y, x+y] = [x, y]+[y, x] = 0$ implies anticommutativity $[x, y] = -[y, x]$.

Let's come back to general linear algebra. Let V be a finite dimensional vector space over a field F, say $\dim V = n$. End$V$ denotes the set of all linear transformations $V \longrightarrow V$. Since End$V$ is isomorphic to the associative algebra of all n by n matrices with entries in F, the dimension of End$V$ is n^2.

Definition 2.1.4. When we define the space End$V$ with Lie bracket, i.e. $[x, y] = xy - yx$, End$V$ is a Lie algebra and we call it a **general linear algebra**, denoted by $gl(V)$. Moreover, any subalgebra is called a **linear Lie algebra**.

There are four families of linear Lie algebra (see [10], (§1.2, p2)). But in this thesis, we are mainly interested in the special linear algebras which are defined as follows.

Definition 2.1.5. A **special linear algebra** is the set of endomorphisms of V having trace zero and denoted by $sl(V)$ or $sl(m + 1)$ where $\dim V = m + 1$.

Suppose the dimension of V is $m + 1$. What is the dimension of $sl(V)$? Let $e_{i,j}$ denote the $(m + 1) \times (m + 1)$ matrix such that the (i, j) entry is 1 and others are zeros. The trace zero property of $sl(V)$ implies that all $e_{i,j}$ where $i \neq j$ and all $h_i = e_{i,i} - e_{i+1,i+1}$ where $1 \leq i \leq m$ form the basis of $sl(V)$. There are $(m + 1)^2 - m + (m - 1) = (m + 1)^2 - 1$ of them in total. For example, the basis of $sl(2)$ is

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

This standard basis of $sl(2)$ will serve us throughout this thesis.

Definition 2.1.6. A subspace I of a Lie algebra L is called an **ideal** of L if $x \in L$, $y \in I$ together imply $[x, y] \in I$.

Definition 2.1.7. The **center** in Lie algebra is defined as $Z = \{z \in L \mid [x, z] = 0$ for all $x \in L\}$.

Definition 2.1.8. Let A and B be two subspaces of Lie algebra L, then $[A, B] = span\{[a, b] \mid \text{for all } a \in A \text{ and all } b \in B\}$.

Definition 2.1.9. The **derived algebra** of a Lie algebra L is defined to be $[L, L]$.

Example 2.1.1. If L is a Lie algebra, 0 and L itself are ideals of L. The center $Z(L)$ is an ideal of L. The derived algebra of L is an ideal.

Definition 2.1.10. Let L be a Lie algebra and $[L, L] \neq 0$, and L has no ideal except 0 and itself. We call L **simple**.

Definition 2.1.11. A linear transformation $\phi : L \longrightarrow L'$, where L and L' are Lie algebras over F, is called a **homomorphism** if $\phi([x, y]) = [\phi(x), \phi(y)]$ where $x, y \in L$.

Definition 2.1.12. Suppose we have a sequence of ideals of L (the derived series) by $L^{(0)} = L,\ L^{(1)} = [L, L],\ L^{(2)} = [L^{(1)}, L^{(1)}]\ , \cdots ,\ L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$. We call L **solvable** if $L^{(n)} = 0$ for some nonnegative integer n.

Remark 2.1.2. Note $(L^{(1)})^{(1)} = [L^{(1)}, L^{(1)}] = L^{(2)}$. In fact, this holds in general:

$$(L^{(i)})^{(j)} = L^{(i+j)}.$$

Example 2.1.2. Commutativity of L implies solvable since $[L, L] = 0$. But simple algebras are nonsolvable since L simple implies $Z(L) = 0$ and $[L, L] = L \neq 0$.

Proposition 2.1.1. [10], (§3.2, p11) Let L be a Lie algebra, then we have

1. If L is solvable, then all subalgebras of L and homomorphic images of L are solvable.

2. If I is a solvable ideal of L such that L/I is solvable, then L is solvable.

3. If I, J are solvable ideals of L, then $I + J$ is solvable.

Proof. See [10], (§3.2, p11). □

Remark 2.1.3. Let L be a Lie algebra. Suppose S is a maximal solvable ideal of L (i.e., one included in no larger solvable ideal) and I is any other solvable ideal of L. The part 3 of above proposition tells that $S + I = S$ since the sum of two ideals is an ideal. This proves the existence of a unique maximal solvable ideal.

Definition 2.1.13. Let L be a Lie algebra. The maximal solvable ideal of L is called **radical** and denoted by $RadL$.

Definition 2.1.14. Let L be a Lie algebra. Then L is **semisimple** if $RadL = 0$.

Many textbooks use another way to define semisimple. It can be proved that they are equivalent. More information can be found in [10] (§5, pp21-22).

Definition 2.1.15. A Lie algebra is **semisimple** if it isomorphic to a direct sum of simple Lie algebras.

In this thesis, we mainly use the first definition of semisimple.

Example 2.1.3. If a Lie algebra L is simple, then it is semisimple since $[L, L] = L$ and 0 is its maximal solvable ideal.

Example 2.1.4. The $sl(2)$ is a simple Lie algebra, i.e., it has no non-trivial ideals. So it is semisimple.

Definition 2.1.16. A **representation** of a Lie algebra L is a homomorphism $\phi : L \longrightarrow gl(V)$. In other words, ϕ is a linear map that satisfies $\phi([x, y]) = \phi(x)\phi(y) - \phi(y)\phi(x)$ where $x, y \in L$.

Sometimes it is convenient to use another way to define the representations.

Definition 2.1.17. Let L be a Lie algebra. A vector space V over F, endowed with an operation $L \times V \longrightarrow V$ (denoted by $(x, v) \longmapsto x.v$ and called x acts on v) is called an **L-module** if the following conditions hold:

(1) $(ax + by).v = a(x.v) + b(y.v)$.

(2) $x.(av + bw) = a(x.v) + b(x.w)$.

(3) $[x, y].v = x.(y.v) - y.(x.v)$ where $(x, y \in L; v, w \in V; a, b \in F)$.

It is not hard to see that definition 2.1.16 and 2.1.17 are equivalent, i.e. if $\phi : L \longrightarrow gl(V)$ is a representation of L, then V can be viewed as an L-module via the action defined to be $x.v = \phi(x)(v)$. On the other hand, given an L-module V, the (3) in definition 2.1.17 defines a representation $\phi : L \longrightarrow gl(V)$.

Definition 2.1.18. A subspace W of L-module V is called L-submodule if $L.W \subseteq W$, where $L.W = \{l.w \mid \text{for all } l \in L \text{ and all } w \in W\}$.

Definition 2.1.19. A **homomorphism of L-modules** is a linear map $\phi : V \longrightarrow W$ such that $\phi(x.v) = x.\phi(v)$.

There is an example of representation which will be recalled in later chapters: the adjoint representation. The adjoint representation $ad : L \longrightarrow ad(L)$ sends x in L to $ad(x)$ where $ad(x)(y) = [x, y], x, y \in L$. It is obvious that adjoint representation is a linear transformation since $ad(x)(my+nz) = [x, my+nz]$ where $x, y, z \in L; m, n \in F$. Moreover, it preserves the Lie bracket:

$$\begin{aligned}
[ad(x), ad(y)](z) &= ad(x)ad(y)(z) - ad(y)ad(x)(z) \qquad (2.1.1)\\
&= ad(x)([y, z]) - ad(y)([x, z])\\
&= [x, [y, z]] - [y, [x, z]]\\
&= [x, [y, z]] + [y, [z, x]]\\
&= -[z, [x, y]] \quad (\textit{by Jacobi identity})\\
&= [[x, y], z].
\end{aligned}$$

Definition 2.1.20. An L-module V is called **irreducible** if it has precisely two L-submodules (itself and 0).

Definition 2.1.21. An L-module V is called **completely reducible** if V is direct sum of irreducible L-submodules.

Theorem 2.1.1. Weyl's Theorem [[10], (§6.3, pp28-29)]
Let $\phi : L \longrightarrow gl(V)$ be a (finite dimensional) representation of a semisimple Lie algebra. Then ϕ is completely reducible.

Proof. See [10], (§6.3, pp28-29). $\square$

2.2 Lie Polynomials

Definition 2.2.1. Let L_0 be a Lie algebra over K, where K is a commutative ring with unit, and A is a set. Let $i : A \longrightarrow L_0$ be a mapping. The Lie algebra L_0 is called **free** on A if for any Lie algebra L and any mapping $f : A \longrightarrow L$, there is a unique Lie algebra homomorphism $\overline{f} : L_0 \longrightarrow L$ such that $f = \overline{f} \circ i$.

Let A be a set, called the alphabet, whose elements are letters. We usually denote a free Lie algebra by $L(A)$ and say $L(A)$ is generated by A. A free Lie algebra is always defined on a generating set.

Definition 2.2.2. A **word**, denoted by w, on A is a finite sequence of elements of A and the empty sequence is called the **empty word**.

Definition 2.2.3. Let w and v be two words, the **concatenation product** of w and v is wv. i.e. if $w = ab$ and $v = ba$, then the concatenation product of w and v is the new word $abba$.

Definition 2.2.4. The set of all words over A is called the **free monoid** on A with the concatenation product, denoted by A^*.

Definition 2.2.5. The **length** of a word is the number of letters in the word (counting repetitions), denoted by $|w|$. If we regard a word as a monomial, then the **degree** of the monomial is identical to its length.

Definition 2.2.6. Let A be a set and K be a commutative ring with unit. A **noncommutative polynomial** on A over K is a linear combination over K of words on A or monomials on A.

The set of all such polynomials is denoted by $K\langle A\rangle$ which has a K-algebra structure. In other words, it is the K-algebra of the free monoid on A. In fact, $K\langle A\rangle$ is the free associative K-algebra generated by A. More properties of $K\langle A\rangle$ can be found at [[24],§1.1].

Definition 2.2.7. A **Lie polynomial** is an element of the smallest K-submodule of $K\langle A\rangle$ containing A and closed under the Lie bracket.

See [24], (§1.2) and [24], (§2.4) for more details about Lie polynomials.

2.3 Subalgebras of Free Lie Algebras

In this section, we review one of the most important and useful theorems in free Lie algebras. This result is due to Shirshov (1953) and Witt (1956).

Definition 2.3.1. A finite family $P_1, \ldots, P_n$ of polynomials in $K\langle A\rangle$ is **dependent** if either some $P_j = 0$ or if there exist polynomials $Q_1, \ldots, Q_n$ such that $\deg(\sum_j P_j Q_j) < \max_j \deg(P_j Q_j)$.

Note: If nonzero polynomials $P_1, \ldots, P_n$ are linearly dependent over K, then they are dependent.

Definition 2.3.2 ([24], (§2.1, p42))**.** A Lie polynomial P is **Lie-dependent** on Lie polynomials $P_1, \ldots, P_n$ if $P = 0$ or if there exists a Lie polynomial $f(x_1, \ldots, x_n)$ in $L(x_1, \ldots, x_n)$ such that $\deg(P - f(x_1, \ldots, x_n)) < \deg(P)$, and that each word in the x_i appearing in f is of some degree d_i in x_i with $\sum d_i \deg(P_i) \leq \deg(P)$.

Lemma 2.3.1 ([24], (§2.1, p42))**.** Let $P_1, \ldots, P_n$ be a dependent family of Lie polynomials, with $\deg(P_1) \leq \ldots \leq \deg(P_n)$. Then some P_i is Lie-dependent on $P_1, \ldots, P_{i-1}$.

Proof. See [24], (§2.1, pp42-45). □

Theorem 2.3.1. [Shirshov-Witt theorem] [[24], (§2.2, pp44-45)]
Each Lie subalgebra of a free Lie algebra is free.

Proof. Let L be a Lie subalgebra of the free Lie algebra $L(A)$. Let E_n be the collection of polynomials of degree less than n

$$E_n = \{P \in L | deg(P) \leq n\}.$$

Let $\langle E \rangle$ denote the Lie subalgebra generated by $E \subset L(A)$. Define

$$E'_n = E_n \cap \langle E_{n-1} \rangle$$

So E'_n is a subspace of E_n. Now we can construct the following chain:

$$\{0\} = E_0 = E'_1 \subseteq E_1 \subseteq E'_2 \subseteq E_2 \subseteq, \ldots, \subseteq E_{n-1} \subseteq E'_n \subseteq E_n \ldots.$$

Let X_n be a subset of E_n which defines a basis of E_n mod E'_n. Define $X = \bigcup_{n \geq 1} X_n$. We show that L is free on X. It is enough to show that L is isomorphic to $L(B)$, where B is an alphabet with a bijection $B \longrightarrow X$, $b \longmapsto x_b$. So it is enough to show the following 2 claims:

(1) X generates L;

(2) For each nonzero Lie polynomial $f(b)_{b \in B} \in L(B)$, one has $f(x_b)_{b \in B} \neq 0$.

For (1). Choose a $P \in L$ such that $deg(P) = n$, so P is in E_n. For some scalars α_x, one has

$$Q = P - \sum_{x \in X_n} \alpha_x x \in E'_n.$$

By the definition of E'_n, Q is in the subalgebra generated by E_{n-1}. By induction, we get Q is also in the subalgebra generated by X. We proved that P is in $\langle X \rangle$, hence X generates L.

For (2). We prove this claim by contradiction. Suppose $f(P_1, \ldots, P_q) = 0$, for some nonzero Lie polynomial $f(b_1, \ldots, b_q) \in L(B)$ and some $P_1, P_2, \ldots, P_q \in X$ with $deg(P_1) \leq \ldots \leq deg(P_q)$. Moreover, there exists a nonzero polynomial f in $K\langle B \rangle$ such that $f(P_1, \ldots, P_q) = 0$. Take such a polynomial with least degree and write it as

$$f = \sum_{i=1}^{q} b_i g_i$$

where g_i is a polynomial. By minimality, some $R_i = g_i(P_1, \ldots, P_q)$ is nonzero. Since

$$0 = f(P_1, \ldots, P_q) = \sum_i P_i R_i.$$

We deduce that $P_1, \ldots, P_q$ are dependent.

By Theorem 2.3.1 some polynomial P_i is Lie-dependent on $P_1, \ldots, P_{i-1}$. This can be rewritten as: P_i plus a linear combination of those P_j, $j < i$, of the same degree as P_i is equal to a Lie expression of the others (which are of degree less than P_i) plus an element of E_{n-1}, with $n = deg(P_i)$. This implies that the polynomials in X_n are not linearly independent mod E'_n, which is a contradiction. □

The proof appears in [24], (§2.2, pp44-45).

2.4 The Hall Basis

A magma is a set with a binary operation, which is not necessarily associative. A basis for a free Lie algebra is the Hall set or the Hall basis, which is a particular kind of subset of the free magma on an alphabet. The free magma $M(A)$ (see [24], p4) over an alphabet A, which is always defined to be a **finite** and **nonempty** set in this thesis, can be identified with a set of binary, complete, rooted trees with leaves labeled by elements in A. Each binary tree of order at least 2 can be written as

$$t = (t^{'}, t^{''})$$

where t' is its immediate left subtree and $t^{''}$ is its immediate right subtree. The binary operation of $M(A)$ is the mapping $M(A) \times M(A) \to M(A)$, $(t^{'}, t^{''}) \to t$. We will see that the relationship between binary trees and Hall words is that we can interpret in the tree t corresponding to a Hall word h each node as Lie bracketing. So every Hall word can be represented by a unique complete binary tree (Hall tree). Hall sets were introduced by the British mathematician Philip Hall (1904 - 1982). It is shown that the commutators which arise in Philip Hall's collecting process [12] will serve as a basis. Ernst Witt (1911-1991) proved that there is an isomorphism between free Lie rings (algebras) and the higher commutator groups of free groups. This isomorphism implies that the problem of finding a basis is the same in both instances. Those commutators are called Hall words in this thesis or standard monomials in [11]. We follow the way (see [11]) introduced by Marshall Hall, who was a student of Philip Hall in Cambridge, to construct the Hall basis recursively from lower degree to higher degree. All definitions and theorems in this section are from the book of Reutenauer (see [24]) and [11].

There is a canonical map from $M(A)$ onto A^* (free monoid over A) defined by $f(a) = a$ if $a \in A$ and $f(t) = f(t')f(t'')$, if $t = (t', t'')$ is of degree greater than 1. Degree (length) of a tree (word) is usually defined to be the number of its leaves (letters).

Definition 2.4.1. We call the $f(t)$ the **foliage** of t.

A Hall set, which will be defined as follows, is a subset of the free magma $M(A)$.

Definition 2.4.2. Let A be a finitely nonempty set and $M(A)$ be the free magma over A. $H \subseteq M(A)$ is called a **Hall set** if all of the following conditions hold:

(1) H has a total order $<_M$;

(2) $A \subseteq H$;

(3) for any tree $X = [X', X'']$ in H, we have $X', X'' \in H$ as well;

(4) for any tree $X = [X', X'']$ in H, $X' >_M X''$;

(5) for any tree $X = [X', X'']$ in H, either $X' \in A$ or $(X')'' \leq_M X''$.

where A is called generating set, the elements in A are called generators, X' denotes the left subtree of X, X'' denotes the right subtree of X and $(X')''$ denotes the right subtree of the left subtree of X.

Remark 2.4.1. The total order in part (1) is not uniquely determined by the other conditions (2)-(5). On the next page, we will define a particular total order that will be used throughout the rest of the thesis.

Definition 2.4.3. The elements in the Hall set are usually called **Hall trees**.

Definition 2.4.4. We call the **Hall word** the foliage of the Hall tree.

Theorem 2.4.1 ([24], §4.2 pp86-89)**.** Given a total order on a Hall set, each Hall word is the foliage of a unique Hall tree.

Proof. See [24], (§4.2 pp86-89). □

In this thesis, we can simply interpret each node in a Hall tree as a Lie bracket. More information about this interpretation can be found at ([24], pp89-90). So we can identify Hall words and Hall trees from now on.

Example 2.4.1. A Hall tree $X = (X', X'')$ of order greater than or equal to 2 can be written as

$$X = [X', X'']$$

where X' is the left subtree and X'' is the right subtree and $[*, *]$ denotes the Lie bracket.

Actually, Hall sets are not unique. A specific Hall set depends on how you define the total order. In this thesis, we use the following method to define the order of the Hall set of 2 generators. (This definition coincides with the first definition in Marshall Hall's original paper ([11], §2, pp576-577).

Let $A = \{a, b\}$, and define a total order on A by setting $a <_A b$. Let $M(A)$ denote the free magma on A; so $M(A)$ is the set of all nonassociative words in $\{a, b\}$. Any $X \in M(A)$ can be written uniquely as $X = [X', X'']$ where $X', X'' \in M(A)$. Now we can define the total order $<_{M(A)}$ or simply $<_M$ on $M(A)$ by making $<_M$ agree with $<_A$ on A (symbol $<_M$ should be read as"precedes"), then for $X, Y \in M(A) \setminus A$, with $Y = [Y', Y'']$, we define $X <_M Y$ if and only if:

(1) $\deg(X) < \deg(Y)$, or

(2) $\deg(X) = \deg(Y)$, but $X' <_M Y'$, or

(3) $\deg(X) = \deg(Y)$ and $X' = Y'$, but $X'' <_M Y''$.

To illustrate the recursive construction of H, all Hall trees (Hall words) of degree ≤ 5 are listed below by total order $<_M$:

1: a_1, b_2,

2: $[b,a]_1$,

3: $[[b,a],a]_1$, $[[b,a],b]_2$,

4: $[[[b,a],a],a]_1$, $[[[b,a],a],b]_2$, $[[[b,a],b],b]_3$,

5: $[[[b,a],a],[b,a]]_1$, $[[[b,a],b],[b,a]]_2$, $[[[[b,a],a],a],a]_3$, $[[[[b,a],a],a],b]_4$, $[[[[b,a],a],b],b]_5$, $[[[[b,a],b],b],b]_6$.

The subscript of each Hall word of degree d denotes its position in the Hall basis of degree d. The Hall words of degree $6, 7, 8, 9$ and 10 are listed in the Appendix C.

Definition 2.4.5. An element of free Lie algebra L will be said to be in **standard form** if it is a linear combination of Hall words.

Now we can ask the following question: If we are unlucky to choose an element in a free Lie algebra L which is not a linear combination of Hall words, then is there a process which can put such element into a standard form in L? Indeed, such a process (Algorithm 2.4.1) exists.

Definition 2.4.6. Let $M(A)$ be the free magma over a set A. Then the linear combination of elements of degree n in $M(A)$ is called a **homogeneous expression of degree** n or a **homogeneous Lie polynomial of degree** n.

Algorithm 2.4.1. Marshall Hall's algorithm for Standard Form[[11]]

- **Input:** Homogeneous expression λ of degree n.

- **Output:** The standard form of λ.

- Step 1. If λ is a generator, then we return λ. Else, we decompose each monomial in λ into two factors as follows: $\lambda = \sum_k t_k[Y_k, Z_k]$ where t_k is a scalar, $Y_k, Z_k \in M(A)$ and $deg(Y_k), deg(Z_k) < n$. We recursively call the Algorithm 2.4.1 to express the Y's and the Z's as linear combinations of Hall words, where $Y_k = \sum_i a_{i,k}U_{i,k}$, $Z_k = \sum_j b_{j,k}V_{j,k}$. Then

$$\sum_k t_k[Y_k, Z_k] = \sum_{i,j,k} t_k a_{i,k} b_{j,k}[U_{i,k}, V_{j,k}].$$

- Step 2.

  1. If $U_{i,k} = V_{j,k}$, then $[U_{i,k}, V_{j,k}] = 0$;
  2. If $U_{i,k} <_M V_{j,k}$, then $[U_{i,k}, V_{j,k}] = -[V_{j,k}, U_{i,k}]$;
  3. If $U_{i,k} >_M V_{j,k}$, then do nothing.

- Step 3. Let $U_{i,k} = [Z, W]$ be the standard form of U, put

  1. If $V_{j,k} \geq_M W$, then $[U_{i,k}, V_{j,k}] = [[Z, W], V_{j,k}]$;
  2. If $V_{j,k} <_M W$, then $[U_{i,k}, V_{j,k}] = -[[W, V_{j,k}], Z] + [[Z, V_{j,k}], W]$. (Jacobi Identity)

We have implemented this algorithm in Maple and it is presented in Appendix A.2.

Now we have a tool which can transfer every homogeneous expression of degree n to a linear combination of Hall words of degree n. Actually, this algorithm tells us that Hall words share certain properties with bases. If we can show that Hall words of degree n are linearly independent, then they actually form the basis of the free Lie algebra of degree n. Moreover, we can regard $L(A) = \bigoplus_{n=1}^{\infty} L_n$ as a graded algebra (A definition of graded algebra can be found in Serge Lang, *Algebra*.) by letting the elements in A to be of degree one, where L_n is the subspace of degree n. The collection of all Hall words in all degrees is the basis for a free Lie algebra $L(A)$ if the Hall words in each degree are linear independent. At last, one of the most famous theorems in free Lie algebra is stated:

Theorem 2.4.2. Marshall Hall[[11]]
The Hall words form a basis of the free Lie algebra $L(A)$ generated by all elements in A.

Proof. The algorithm has shown that every homogeneous expression of degree n can be written as a linear combination of Hall words. See [11], (§3) for the proof of the linear independence of Hall words. □

Recall the Definition 2.2.6 of Lie polynomials, in the free associative algebra it is easy to see that if we expand the Lie bracket of a Hall word, we obtain a Lie polynomial. This is because Lie polynomials are elements of a free Lie algebra and Hall words form the basis for a free Lie algebra.

Definition 2.4.7. A Hall word is denoted by X_i^j where i is the degree and j is the position of the Hall word in an ordered basis of L_i. See Appendices C and D.

Example 2.4.2. $[[[[b,a],a],a],a]$ is the third Hall word in the ordered set L_5. So we use X_5^3 to represent it.

2.5 The Witt Formula

In this section, let $L = \bigoplus_{n=1}^{\infty} L_n$ denote the free Lie algebra with d_i generators of degree i where d_i is a positive integer and i is a nonnegative integer. The total number $\sum_i d_i$ of generators may be finite or infinite. We look for a formula which computes the dimension of L_n. Witt obtained the following formula which is in the simplest case $d_1 = d$ and $d_i = 0$ for $i \geq 2$.

Definition 2.5.1. The **Mobius function** is the function μ defined by
$\mu : \mathbb{N} \longrightarrow \{0, 1, -1\}$ such that

1. $\mu(n) = 1$ if $n = 1$;
2. $\mu(n) = 0$ if $p^2 | n$ for some prime p;
3. $\mu(n) = (-1)^t$ if $n = p_1 p_2 \cdots p_t$ where $p_1, \cdots, p_t$ are t distinct primes.

Theorem 2.5.1. Principle of Mobius Inversion [[25], (§3.7)]
If $f(n)$ and $g(n)$ both are functions such that $f, g : \mathbb{N} \longrightarrow \mathbb{C}$ and

$$g(n) = \sum_{d|n} f(d)$$

for every integer $n \geq 1$, then we have

$$f(n) = \sum_{d|n} \mu(d) g(\frac{n}{d})$$

for every integer $n \geq 1$.

Proof. See [25], (§3.7, p116). □

Theorem 2.5.2. The Poincare-Birkhoff-Witt theorem (PBW) [[24], (pp1-7)] The free associative algebra is the universal enveloping algebra of the free Lie algebra.

Another reference about PBW is [2], (§2.3.2, pp49-50).

Theorem 2.5.3 ([2], §3.1, pp74-75)**.** Let $L(A)$ be the free Lie algebra over $\mathbb{C}$ with d generators and n denote the length of Hall words. Then the dimension of the space of homogeneous expression of degree n or the number of Hall words in Hall basis is:

$$L_d(n) = \frac{1}{n} \sum_{m|n} \mu(m) d^{n/m}. \tag{2.5.1}$$

Proof. Let $\{X_1, X_2, X_3, \ldots\}$ be a basis in $L(A)$ and the degree of X_i is equal to d_i. According to PBW, $L(A)$ is embedded in $K\langle A\rangle$ (recall $K\langle A\rangle$ is a free associative algebra over A) and a basis of $K\langle A\rangle$ can be chosen in either of the two following ways.

(1) The set of all words of length n in A; there are d^n of them.

(2) The set of all products of the form

$$X_s^{e_s} X_{s-1}^{e_{s-1}} \ldots X_1^{e_1}$$

where e_i are some non-negative numbers and $n = e_1 h_1 + \ldots + e_s h_s$.

Thus h_n is equal to the number of all finite sequences $(e_1, \ldots, e_n)$ such that

$$n = \sum_{i=1}^{s} e_i h_i.$$

In other words, the coefficients of the following two power series are the same:

$$A(t) = \prod_{i=1}^{\infty} \frac{1}{1 - t^{h_i}}, \qquad B(t) = \frac{1}{1 - dt}.$$

Also,

$$A(t) = \prod_{i=1}^{\infty} \frac{1}{1 - t^{h_i}} = \prod_{i=1}^{\infty} (1 + t^{h_i} + t^{2h_i} + \ldots)$$

and

$$B(t) = \frac{1}{1-dt} = 1 + dt + d^2t^2 + \ldots.$$

Now the number of degree h_i equals to the same number m is precisely the dimension $L_d(m)$. Hence

$$\prod_{m=1}^{\infty} \frac{1}{(1-t^m)^{L_d(m)}} = \frac{1}{1-dt}.$$

Taking logarithm to right hand side, we obtain

$$\log \frac{1}{1-dt} = \log((1-dt)^{-1}) = -\log(1-dt) = -\sum_{n=1}^{\infty} \frac{1}{n}(-1)^n d^n t^n.$$

Taking logarithm to left hand side, we obtain

$$\begin{aligned}
\log \prod_{m=1}^{\infty} \frac{1}{(1-t^m)^{L_d(m)}} &= \sum_{m=1}^{\infty} \log(1-t^m)^{-L_d(m)} = \sum_{m=1}^{\infty} -L_d(m)\log(1-t^m) \\
&= \sum_{m=1}^{\infty} -L_d(m) \sum_{v=1}^{\infty} \frac{1}{v}(-t)^{mv} \\
&= \sum_{m,v} \frac{1}{v} - L_d(m)(-t)^{mv}.
\end{aligned}$$

Combine above 2 equations, we obtain

$$\sum_{m,v} \frac{1}{v} L_d(m) t^{mv} = \sum_{n=1}^{\infty} \frac{1}{n} d^n t^n$$

Hence

$$\sum_{mv=n} \frac{1}{v} L_d(m) = \frac{1}{n} d^n.$$

Finally we have

$$d^n = \sum_{m|n} m L_d(m). \tag{2.5.2}$$

By principle of Mobius inversion, we obtain

$$L_d(n) = \frac{1}{n} \sum_{m|n} \mu(m) d^{n/m}.$$

□

The proof appears in [2], (§3.1, pp74-75). You can also find a proof in [24], (§4.2, pp96-97).

Remark 2.5.1. Given a number of generators $d_1 = d$, we can use formula (2.5.1) to compute the number of Hall words of degree i for $i \geq 2$. But this formula only works for $d_1 = d$ and $d_i = 0$ for $i \geq 2$. We need a more general formula which can compute the free Lie algebra generated by d_i generators of degree i where d_i is a positive integer and i is a nonnegative integer.

Theorem 2.5.4. Generalized Witt Formula [[19], (Theorem 1.1)]
Let $V = \bigoplus_{i=1}^{n} V_i$ be a graded vector space over a field of characteristic 0, and let $L = \bigoplus_{i=1}^{n} L_i$ be the free Lie algebra generated by V. Then we have

$$\dim L_n = \frac{1}{n}\sum_{f|n}\mu(f)W(\frac{n}{f}), \quad where \;\; W(p) = p\sum_{j=1}^{p}\frac{1}{j}\sum_{i_1+\cdots+i_j=p} d_{i_1}\cdots d_{i_j} \tag{2.5.3}$$

where the inner sum in $W(p)$ is over all j-compositions of p, i.e. all ordered partitions of p into j parts.

Proof. A proof can be found in [19], (pp566-567) and [15]. □

Remark 2.5.2. Theorem 2.5.3 is a special case of Theorem 2.5.4 where $d_1 = 2$ and $d_i = 0$ for all $i > 1$. The MAPLE implementation is in Appendix A.3.

2.6 Multiplicity Formula

Let L_n denote subspace of the free Lie algebra L consisting of all homogeneous Lie polynomials of degree n and let K_n denote the subspace of K consisting of the homogeneous invariants of degree n where K is a subalgebra of L of Lie invariants (for $sl(n)$ in the natural representation). Let M_n denote the module over the symmetric group S_n spanned by the multilinear Lie polynomials of degree n. So M_n is a subspace of the free Lie algebra on n generators. The dimension of K_{qm}, where q is the number of generators of free Lie algebra L, equals the multiplicity, in the S_{qm}-module M_{qm}, of the simple S_{qm}-module corresponding to the partition m^q. So we obtain following proposition.

Theorem 2.6.1 ([24], (pp206-207)). The multiplicity in M_n of the simple S_n-modulo corresponding to the partition λ is given by the formula

$$\alpha_\lambda = \frac{1}{n}\sum_{f|n}\mu(f)\chi^{\lambda}_{f^{n/f}}$$

where μ is the Mobius function, and $\chi^{\lambda}_{\lambda'}$ denotes the (λ, λ') entry of the character table for S_n.

Example 2.6.1. The dimension of K_{2m}, where we consider a free Lie algebra generated by 2 elements, equals the multiplicity, in the S_{2m}-module M_{2m}, of the simple S_{2m}-module corresponding to the partition m^2. In other words, the dimension is equal to the number of standard tableaux of shape m^2 and major index congruent to 1 mod $2m$. Therefore, we obtain

$$\dim K_{2m} = \frac{1}{2m}\sum_{f|2m}\mu(f)\chi^{m^2}_{f^{2m/f}}.$$

The MAPLE implementation for this formula is in Appendix A.3.

Degree:	2	4	6	8	10	12	14	16	18	20
Dimension of Invariants space:	1	1	1	5	9	33	85	276	827	2693

Table 2.1: Dimension table for example 2.6.1

2.7 Invariant Lie Polynomials

We discuss the 2 generator case in this section, but all terminology applies equally well to any finite number of generators. Let L (or $L(A)$) denote the free Lie algebra on generators $A = \{a, b\}$ over a complex field. We regard $L = \bigoplus_{n=1}^{\infty} L_n$ as a graded Lie algebra by letting $deg(a) = deg(b) = 1$. Let's consider the natural action of $sl(2)$ on the free Lie algebra generated by $A = \{a, b\}$, and the fixed points set K which are invariant under the natural actions, form a Lie subalgebra of L, which is also free by Shirshov-Witt Theorem. The free Lie algebra K is known as the algebra of Lie invariants or invariant Lie polynomials.

Let K_n denote the subspace of K consisting of the homogeneous Lie invariants of degree n. Wever proved that the degree d of a Lie invariant must be a multiple of the number of generators (see [26]). We have

$$K_n = \{0\} \text{ for } n \text{ odd}$$

and that $dim(K_2) = 1$, $dim(K_4) = 0$, $dim(K_6) = 1$ and $dim(K_8) = 1$. The Lie invariants of degree 2, 6 and 8 are

$$I_2 = [b, a] \quad \text{and} \quad I_6 = [[[b, a], b], [[b, a], a]] \quad \text{and} \quad I_8 = [I_6, I_2].$$

Definition 2.7.1. An **inner product space** is a vector space V over a field F together with an inner product $\langle * \mid * \rangle : V \times V \to F$ which satisfies the following conditions for all vector $v_1, v_2, v_3 \in V$ and $a \in F$.

(1). $\langle v_1 + v_2 \mid v_3 \rangle = \langle v_1 \mid v_3 \rangle + \langle v_2 \mid v_3 \rangle$.

(2). $\langle av_1 \mid v_2 \rangle = a\langle v_1 \mid v_2 \rangle$.

(3). $\langle v_1 \mid v_2 \rangle = \overline{\langle v_2 \mid v_1 \rangle}$, the bar denoting complex conjugation.

(4). $\langle v_1 \mid v_1 \rangle > 0$ for all nonzero $v_1 \in V$.

Definition 2.7.2. Let V be an inner product space and S any set of vectors in V. The **orthogonal complement** of S is the set $S^{\perp}$ of all vectors in V which are orthogonal to every vector in S.

Recall that the subalgebra K of free Lie algebra L is also free. We are primarily interested in the primitive invariants in K, i.e. a set of free generators for K. Let J_n denote the subspace of primitive invariants in K_n. Precisely, we take J_n to be the orthogonal complement in K_n of the span of the non-primitive invariants in K_n. All $\dim L_n$, $\dim K_n$ and $\dim J_n$ can be computed by the Witt formula. The $\dim L_n$ can be computed by Theorem 2.5.3 with $d = 2$ or Theorem 2.5.4 with $d_1 = 2$ and $d_i = 0$ for all $i > 1$. We can use Theorem 2.6.1 to obtain $\dim K_n$ directly. Computing $\dim J_n$ is not that obvious. Suppose we already know $\dim J_k$ for $k < m$, we can generalized Witt formula (Theorem 2.5.4) to compute K'_n which denotes the free Lie algebra on $\dim J_k$ generators of degree k for $k < m$. Then we obtain

$$\dim J_n = \dim K_n - \dim K'_n.$$

Now, we can compute $\dim L_n$, $\dim K_n$ and $\dim J_n$ in natural representation L of $sl(2)$ or $sl(3)$ where L is a free Lie algebra generated by a, b or a, b, c respectively by those formulas introduced in last section. The table of dimension of L_n, K_n, J_n for $d_1 = 2$ and 3 is listed in Appendix B.

2.8 The Hermite Normal Form

In linear algebra, the row echelon form is used for matrices over a field and the Hermite normal form is used for matrices over a PID (principal ideal domain), in particular, over $\mathbb{Z}$ in this thesis. The Hermite normal form plays an important role in the study of many subfields of algebra. i.e. for an integer matrix A, we can use Hermite normal form to find a integral basis for $nullspace(A)$.

Definition 2.8.1. An integer square matrix with a determinant of $+1$ or -1 is called **unimodular transform matrix**.

Definition 2.8.2. For any $m \times n$ matrix A, $A[i_1 \dots i_2, j_1 \dots j_2]$ denotes the submatrix of A consisting of row i_1 to row i_2 and column j_1 to column j_2 where $1 \leq i_1 \leq i_2 \leq m$ and $1 \leq j_1 \leq j_2 \leq m$.

Definition 2.8.3. The $m \times n$ matrix H over $\mathbb{Z}$ is in **Hermite normal form (HNF)** if there exists an integer r (the rank of H) with $0 \leq r \leq m$ and a sequence $1 \leq j_1 < j_2 < \cdots < j_r \leq n$ of integers such that:

1. $H_{i,j} = 0$ for $1 \leq i \leq r$ and $1 \leq j < j_i$, (H_{i,j_i} is the first nonzero entry in row i)

2. $H_{i,j_i} \geq 1$ for $1 \leq i \leq r$,

3. $0 \leq H_{k,j_i} < H_{i,j_i}$, for $1 \leq i \leq r$ and $1 \leq k < i$, (the pivot element is the greatest along its column and the elements above are nonnegative)

4. $H_{i,j} = 0$ for $r + 1 \leq i \leq m$ and $1 \leq j \leq n$.

Example 2.8.1. Let A be a 4×6 integer matrix

$$A = \begin{bmatrix} -7 & 7 & 0 & 5 & 1 & -2 \\ 5 & -9 & 0 & 4 & 6 & 7 \\ 4 & -6 & 0 & -9 & -1 & -5 \\ 7 & -7 & 0 & 2 & 6 & 3 \end{bmatrix}.$$

Then the Hermite normal form H of A is

$$H = \begin{bmatrix} 1 & 1 & 0 & 0 & 388 & -475 \\ 0 & 2 & 0 & 0 & 154 & -190 \\ 0 & 0 & 0 & 1 & 87 & -105 \\ 0 & 0 & 0 & 0 & 602 & -736 \end{bmatrix}.$$

Theorem 2.8.1. (Hermite)[[1]]
If A is an $m \times n$ integer matrix, then there exists a unique $m \times n$ integer matrix H in Hermite normal form and an $m \times n$ unimodular transform matrix U such that $UA = H$.

Proof. See [1] (§5.2). □

Remark 2.8.1. Since U is a product of a sequence of elementary matrices, U is not unique in general. It depends on the row operations used to compute H.

Algorithm 2.8.1. Hermite Normal Form Computing Algorithm [[5]]

- Input: An $m \times n$ integer matrix A (no restriction on A).
- Output: The $m \times n$ integer matrix H in Hermite normal form and an $m \times m$ unimodular U such that $UA = H$.

1. Set $H \longleftarrow A$, $U \longleftarrow I_m$, $j \longleftarrow 1$.
2. While $i \leq m$ and $j \leq n$ do:
   if **zerocolumn**(H, i, m, j) then
   (a) Set $j \longleftarrow j + 1$

else

(b) While not $[H_{i,j} > 0$ and **zerocolumn**$(H, i+1, m, j)]$ do:

i. Set $s \longleftarrow \min\{|H_{p,j}| : H_{p,j} \neq 0, i \leq p \leq m\}$.

ii. Choose p so that $|H_{p,j}| = s$.

iii. Exchange rows i and p of H, and the same for U.

iv. If $H_{i,j} < 0$ then multiply row i by -1, and the same for U.

v. For $k = i+1$ to m do:

- Compute q where $H_{k,j} = qH_{i,j} + r$ uniquely with $0 \leq r < H_{i,j}$.
- If $q \neq 0$ then add $-q$ times row i of H to row k, and the same for U.

(c) For $k = 1$ to $i-1$ do:

i. Compute q where $H_{k,j} = qH_{i,j} + r$ uniquely with $0 \leq r < H_{i,j}$.

ii. If $q \neq 0$ then add $-q$ times row i of H to row k, and the same for U.

(d) Set $i \longleftarrow i+1$ and $j \longleftarrow j+1$.

- **zerocolumn**(H, i_1, i_2, j): If $H_{i,j} = 0$ for $i_1 \leq i \leq i_2$ then **true**, otherwise **false**.

Remark 2.8.2. The idea of Algorithm 2.8.1 is a variant of Gaussian elimination. Unfortunately, although the algorithm performs a polynomial number of arithmetic operations, the size of the numbers in the intermediate computations do not have a polynomial upper bound. It may grow exponentially. Looking for the best Hermite normal form computing algorithm is a very active research area in computer algebra at present. Most recent algorithms are polynomial both in number of arithmetic operations and space usage. But they are more complicated. Since the matrices we deal with in the following chapters consist of zero and small integers, Algorithm 2.8.1 is good enough to solve those cases in a reasonable amount of time.

The following useful theorem and its proof are basically from chapter 15, lecture notes on *Lattice Basis Reduction* by Bremner (2009) or [8].

Theorem 2.8.2. Let A be an $m \times n$ integer matrix, let H be the Hermite normal form of A^T, and let U be an $n \times n$ unimodular such that $UA^T = H$. If r is the rank of H, then the last $n - r$ rows of U form a basis for nullspace(A).

Proof. Since $UA^T = H$ and the last $n - r$ rows of H are zero, we have $VA^T = O$ where V is the $(n - r) \times n$ matrix consisting of the last $n - r$ rows of U. Hence $AV^T = O$ which implies the rows of V are in nullspace(A). It remains to show that any vector $X \in$ nullspace(A) is an integer linear combination of the last $n - r$ rows of U. Suppose that $XA^T = O$ for some row vector $X \in \mathbb{Z}^n$, and set $Y = XU^{-1}$. Then we have

$$YH = (XU^{-1})(UA^T) = XA^T = O.$$

Now we can solve the linear system $YH = 0$ from left to right for the components of the row vector Y. Since H is in Hermite normal form, we find that the first r component of Y are zero, and the last $n - r$ components of Y are arbitrary:

$$YH = [y_1, \cdots, y_r, y_{r+1}, \cdots, y_n] \begin{bmatrix} h_{1,1} & h_{1,2} & \cdots & h_{1,j_r} & \cdots & h_{1,m} \\ 0 & h_{2,2} & \cdots & h_{2,j_r} & \cdots & h_{2,m} \\ 0 & 0 & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & h_{r,j_r} & \cdots & h_{r,m} \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} = [0, \cdots, 0].$$

It follows that the last $n - r$ standard unit vectors in $\mathbb{Z}^n$ form a basis for the solution space of $YH = O$. Since $X = YU$, the vector X is an integer linear combination of the last $n - r$ rows of U. This completes the proof. $\square$

We will see in the next chapter that if we want to find Lie invariants, we need to determine the nullspace of an integer matrix which represent the actions by elements of a Lie algebra. Since a basis of the kernel of a given integer matrix can be computed with the Hermite normal form, Theorem 2.8.2 actually tells us a basis for the kernel. The MAPLE implementation of Algorithm 2.8.1 is in Appendix A.4.

Chapter 3

Invariants in the Natural Representation of $sl(2)$

In this chapter, we focus on computing the Lie invariants for the natural representation of $sl(2)$; in this case L is a free Lie algebra generated by 2 variables. First, we use the method developed in [3] to compute the Lie invariants of degree less than 10. Then we review the Lie invariants of degree 10 (this result appears in [3]). After that, we compute the Lie invariants of degree 12 and degree 14. These are original contributions in this thesis.

3.1 Representations of $sl(2)$

We begin to review some useful results about the representations of $sl(2)$. In this section, definitions, lemmas and propositions are from [10] and [17]. We define the ordered standard basis for $sl(2)$ as follows:

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then

$$[h, x] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = 2x,$$

$$[h, y] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} = -2y,$$

$$[x, y] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = h.$$

Simply, we have following equations:

$$[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h.$$

Definition 3.1.1. Let V be a finite-dimensional vector space. If W is any subspace of V, then there exists a subspace W' such that $V = W \bigoplus W'$. The W' is called **complementary** space to W.

Definition 3.1.2. Let V be a finite-dimensional vector space over a field F, and let T be a linear operator on V. We say that T is **semi-simple** if every T-invariant subspace has a complementary T-invariant subspace, where a subspace U is T-invariant if $T.U \subset U$.

Proposition 3.1.1. If T is a linear operator on a finite-dimensional vector space over an algebraically closed field, then T is semi-simple if and only if T is diagonalizable.

Proof. See [17], (§7.5, p264). □

Let's come back to representations of $sl(2)$. Since h is diagonalizable, it's semi-simple. The Jordan decomposition ([17], (§6.4, p29)) tells us that h acts diagonally on V.

Definition 3.1.3. Let V be a finite-dimensional representation of $sl(2)$ over the field F. The eigenspace $V_\lambda = \{v \in V \mid h.v = \lambda v\}$, where $\lambda \in F$, is called a **weight space**, λ is called a **weight** of h and v is called a **weight vector**.

This implies that there is a decomposition of V as direct sum of eigenspaces $V_\lambda = \{v \in V \mid h.v = \lambda v\}$, where $\lambda \in F$.

Lemma 3.1.1. If $v \in V_\lambda$, then $x.v \in V_{\lambda+2}$ and $y.v \in V_{\lambda-2}$.

Proof. Suppose ϕ is the corresponding representation. Then

$$\begin{aligned}[h, x].v = \phi([h, x])v = [\phi(h), \phi(x)]v = (\phi(h)\phi(x) - \phi(x)\phi(h))v \\ = \phi(h)\phi(x)v - \phi(x)\phi(h)v = h.(x.v) - x.(h.v)\end{aligned}$$

So we get $h.(x.v) = [h, x].v + x.(h.v) = 2x.v + wx.v = (\lambda + 2)x.v$ since $[h, x] = 2x$ and $h.v = \lambda v$, and a similar argument for y. □

This lemma is very useful in the problem considered in this thesis. It is the key to find the invariant Lie polynomials. Since $\dim(V)$ is finite, the previous lemma tells us that there must exist $V_\lambda \neq 0$ such that $V_{\lambda+2} = 0$. This helps us to define:

Definition 3.1.4. Any nonzero vector v in V_λ is called a **maximal vector** of weight λ in V_λ if $x.v = 0$.

Definition 3.1.5. Suppose V is an irreducible L-module and v is a maximal vector in V with nonnegative weight. Then the weight of the maximal vector v is called the **highest weight** of V.

Next, we review an example about the representation of $sl(2)$ which will serve us in Chapter 4:

Example 3.1.1. The 3-dimensional adjoint representation of $sl(2)$.

Suppose $\{x, h, y\}$ is the ordered basis of $sl(2)$. There are some relations under Lie bracket of these bases: $ad(x)(x) = [x, x] = 0$, $ad(x)(h) = [x, h] = -2x$ and $ad(x)(y) = [x, y] = h$. Since $ad(x)$ is a linear transformation, the matrix representation of this transformation is

$$ad(x) = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Applying similar arguments to $ad(y)$ and $ad(h)$, we obtain $ad(h)(x) = [h, x] = 2x$, $ad(h)(h) = [h, h] = 0$ and $ad(h)(y) = [h, y] = -2y$. And $ad(y)(x) = [y, x] = -h$ and $ad(y)(h) = [y, h] = 2y$ and $ad(y)(y) = 0$. The matrix representations for those two transformations are:

$$ad(h) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad \text{and} \quad ad(y) = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

Finally, we review some classical results about the irreducible representation which will serve us later. Assume that V is an irreducible L-module. Choose a maximal vector in V, say v_0, and set $v_{-1} = 0$, and

$$v_i = (1/i!)y^i.v_0$$

where $i \geq 0$.

Lemma 3.1.2 ([10], (§7.2, pp32))**.** If v_0 has weight λ then

(a). $h.v_i = (\lambda - 2i)v_i$,

(b). $y.v_i = (i+1)v_{i+1}$,

(c). $x.v_i = (\lambda - i + 1)v_{i-1}$.

Proof. (a). By lemma 3.1.1, we know y brings an element in a weight space with weight λ to a weight space with weight $\lambda - 2$. Since y acts on v_0 i times, the weight of v_i is $\lambda - 2i$. (b). It's just the definition of v_i. (c). We use induction on i. When $i = 0$, we have $x.v_0 = 0$ since v_0 is maximal vector and $v_{-1} = 0$. Observe that

$$\begin{aligned}
ix.v_i &= x.(y.v_{i-1}) \quad (Modification\ of\ the\ definition\ of\ v_i) \\
&= [x,y].v_{i-1} + y.(x.v_{i-1}) \\
&= h.v_{i-1} + y.(x.v_{i-1}) \\
&= (\lambda - 2(i-1))v_{i-1} + (\lambda - i + 2)y.v_{i-2} \quad (Induction\ step) \\
&= (\lambda - 2i + 2)v_{i-1} + (i-1)(\lambda - i + 2)v_{i-1} \\
&= i(\lambda - i + 1)v_{i-1}.
\end{aligned}$$

Divide both sides by i to complete the proof. □

Definition 3.1.6. For any nonnegative integer n, there is an irreducible representation of $sl(2)$ containing a nonzero highest weight vector v_n satisfying the conditions:

$$h.v_n = nv_n, \qquad x.v_n = 0.$$

This representation is unique up to isomorphism of $sl(2)$-modules. It is denoted by $V(n)$ and is called the **representation with highest weight** n. Every finite-dimensional irreducible representation of $sl(2)$ is isomorphic to $V(n)$ for some $n\mathbb{Z}$ and $n \geq 0$.

Lemma 3.1.3 ([4], (§5.3, p59))**.** Let M be a finite dimensional $sl(2)$-module. For every integer n, we define $M_n = \{v \in M \mid h.v = nv\}$ which denotes the subspace of M consisting of all vectors of weight n. Then for any nonnegative integer n, the multiplicity of $V(n)$ as a direct sum of M equals $dimM_n - dimM_{n+2}$.

Proof. See [4], (§5.3, p59). □

3.2 The Number of Lie Invariants

In [26] and [27], Wever showed that the degree n of a Lie invariant must be a multiple of r where r is the number of free generators. He gave a formula for the number of Lie invariants when $r = 2$. See ([24], §8.6, pp206-209). Wever also proved that there exists no Lie invariant of degree 6 for $r = 3$ and there are exactly 4 primitive Lie invariants of degree 9 for $r = 3$.

In [4], Burrow showed the existence of Lie invariants of degree mr for all integers $m > 1$ and $r > 1$ and except in the cases $r = 2$, $m = 2$ and $r = 3$, $m = 2$. In [6], Theorem 2 shows we can construct a Lie invariant for any $q > 3$ and any degree mq where $m \geq 2$. In [7], Burrow established an explicit formula for the number of Lie invariants of every degree $n = mr$. Applying Theorem 2.5.4 to 3 generators case, we have

$$dimK_{3m} = \frac{1}{3m} \sum_{f|3m} \mu(f) \chi^{m^3}_{f^{3m/f}}$$

where $\chi^{m^3}_{f^{3m/f}}$ denotes the $(m^3, 3m/f)$ entry of the character table for S_{3m}. We have a MAPLE implementation in Appendix A.3. In particular, for a free Lie ring (algebra) with 3 generators there are 35 and 398 Lie invariants in degree 12 and degree 15 respectively.

S. J. Kang introduced the generalized Witt formula [19] which was discussed in §2.5 (Theorem 2.5.3). Combining this with the formula developed by Burrow, we can determine the number of primitive Lie invariants in the natural representation of $sl(n)$.

3.3 Degrees 1-9

We begin this section with a discussion about how to compute Lie invariants. In this section, we only consider the natural action of $sl(2)$.

We consider the free Lie algebra L generated by a and b. L_1 is a 2-dimensional vector space with ordered basis $\{a, b\}$ since Hall words a and b generate this space. The coordinate of a is $[1, 0]^T$ and the coordinate of b is $[0, 1]^T$. For convenience, we regard $a = [1, 0]^T$ and $b = [0, 1]^T$. Similar arguments apply to 3 generators case in later chapers. The natural actions of $sl(2)$ on L_1 are

$$x.a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0, \qquad x.b = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a,$$
$$y.a = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = b, \qquad y.b = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0,$$
$$h.a = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a,$$
$$h.b = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -b.$$

Simply, we have

$$x.a = 0, \quad x.b = a, \quad y.a = b, \quad y.b = 0, \quad h.a = a, \quad h.b = -b. \tag{1}$$

We regard L_1 as a copy of the simple highest weight $sl(2)$-module with highest weight 1. This action extends to all of L by linearity and the Leibniz rule for derivations. For example:

$$x.[a, b] = [x.a, b] + [a, x.b] = [0, b] + [a, a] = 0 + 0 = 0.$$

Analysis of the main problem

Now the main problem is to determine what linear combinations of the Hall words X_j^i are invariant under the natural action by $sl(2)$, where j is the degree of Hall words and i is the position of the Hall words in its ordered basis. See Appendices C and D. That is, we want to find elements $Z \in L_j$ which satisfy $D.Z = 0$ for all $D \in sl(2)$, where L_j denotes the subspace of L consisting of all homogeneous Lie polynomials of degree j. So it suffices to consider how the basis of $sl(2)$ acts on L_j. Since L_j is finite dimensional and $sl(2)$ is a semisimple Lie algebra, it can be decomposed as a direct sum of simple highest weight modules by Weyl's Theorem (§2.1). If we want to find Z such that $D.Z = 0$, then Z must be in the weight space of weight 0 (i.e., weight space $L_j^0 = \{Z \in L_j \mid h.Z = 0Z = 0\}$). So if $D \neq h$, then $D = x$ or $D = y$ and by Lemma 2.1.1, action by x brings L_j^0 to L_j^2 and action by y brings L_j^0 to L_j^{-2}. We need to determine Z such that $x.Z = 0$ and $y.Z = 0$. Moreover, if $h.Z = 0$ and $x.Z = 0$, then Z spans a highest weight module with highest weight 0, so it generates a 1-dimensional simple module which implies $y.Z = 0$. So in practice, it suffices to find the Z such that $x.Z = 0$ and $h.Z = 0$. Moreover, Lemma 3.1.2 implies that the x mapping is surjective since parts (b) and part (c) of Lemma 3.1.2 together imply $x.(y.v) = av$ for some nonzero scalar a.

Definition 3.3.1. Let L_i^j be a weight space spanned by all Hall words of degree i with weight j.

Lemma 3.3.1. Let L_i^j be the weight space spanned by all Hall words of degree i with weight j, then

$$L_i^j = \{Z \in L_i \mid h.Z = jZ\} = span\{X_i^k \mid \#a(X_i^k) - \#b(X_i^k) = j\},$$

where $\#a(X_i^k)$ and $\#b(X_i^k)$ denote the number of a in X_i^k and the number of b in X_i^k.

Proof. Recall that $h.a = a$ and $h.b = -b$ and those actions satisfy Leibniz rule for derivations. Suppose $X_i^k = [w_1, w_2, \cdots, w_i]$ where we ignore the Lie brackets except the outermost one and $w_n \in \{a, b\}$ for $n = 1, 2, \cdots, i$. So

$$h.X_i^k = [h.w_1, w_2, \cdots, w_i] + [w_1, h.w_2, \cdots, w_i] + \cdots + [w_1, w_2, \cdots, h.w_i]. \quad (3.3.1)$$

Since Lie bracket is bilinear and $h.w_n = w_n$ if $w_n = a$ and $h.w_n = -w_n$ if $w_n = b$, we can move the constant, 1 if $w_n = a$ and -1 if $w_n = b$, outside of the Lie bracket.

After collecting all terms on the right hand side of (3.3.1), we obtain

$$h.X_i^k = jX_i^k = (\#a(X_i^k) - \#b(X_i^k))X_i^k.$$

We complete the proof. $\square$

Definition 3.3.2. Let L_i^j be the weight space spanned by all Hall words of degree i with weight j. Let B_i^j denote the set consisting of all integers k for which $X_i^k \in L_i^j$.

Example 3.3.1. For $i = 3$, $[[b, a], b] \in L_3^{-1}$ since the number of as is 1 and the number of bs is 2. Moreover, since $[[b, a], b]$ is the second element in the ordered basis of L_3, $2 \in B_3^{-1}$.

We notice that computing B_i^0 and B_i^2 is a crucial step for Lie invariants searching in natural representation of $sl(2)$ since the action of x maps the weight 0 space into weight 2 space. We list integer sets B_i^0 and B_i^2 for $i = 1$ to 9 as follows:

- For $i = 1$, we have $B_1^0 = \varnothing, \quad B_1^2 = \varnothing$.
- For $i = 2$, we have $B_2^0 = \{1\}, \quad B_2^2 = \varnothing$.
- For $i = 3$, we have $B_3^0 = \varnothing, \quad B_3^2 = \varnothing$.
- For $i = 4$, we have $B_4^0 = \{2\}, \quad B_4^2 = \{1\}$.
- For $i = 5$, we have $B_5^0 = \varnothing, \quad B_5^2 = \varnothing$.
- For $i = 6$, we have $B_6^0 = \{1, 3, 7\}, \quad B_6^2 = \{2, 6\}$.
- For $i = 7$, we have $B_7^0 = \varnothing, \quad B_7^2 = \varnothing$.
- For $i = 8$, we have $B_8^0 = \{2, 5, 6, 11, 12, 17, 21, 27\}, \quad B_8^2 = \{1, 4, 9, 10, 16, 20, 26\}$.
- For $i = 9$, we have $B_9^0 = \varnothing, \quad B_9^2 = \varnothing$.

From the above list, we notice that when i is odd, there is no element of weight 0. So there is no Lie invariant in those cases. In fact, this is true for all odd integer i in natural representation of $sl(2)$. When $i = 2$, $|B_2^0| = 1$ and $|B_2^2| = 0$ implies $X_2 = [b, a]$ is a Lie invariant.

Example 3.3.2. By definition, we know the weight of $[[b,a],a]$ is $2-1=1$. We can check this in following way

$$\begin{aligned} h.[[b,a],a] &= [h.[b,a],a] + [[b,a],h.a] \\ &= [[h.b,a]+[b,h.a],a] + [[b,a],h.a] \\ &= [[-b,a]+[b,a],a] + [[b,a],a] \\ &= [0,a] + [[b,a],a] \\ &= 1 \cdot [[b,a],a]. \end{aligned}$$

The number of a in $[[b,a],a]$ is 2 and the number of b in $[[b,a],a]$ is 1, so $[[b,a],a]$ should be in L_3^1. Same argument shows that $[[b,a],b]$ is in L_3^{-1}. There is no weight space of weight 0 since $[[b,a],a]$ is in L_3 which is of odd degree.

Theorem 3.3.1. Any nonzero scalar multiple of $[b,a]$ is a primitive Lie invariant of degree 2 in the natural representation L of $sl(2)$ where L is the free Lie algebra generated by a, b.

Proof. We don't have to prove it in a general way at this time since L_2 is 1-dimensional space. It suffices to check whether $[b,a]$ is an invariant polynomial.

$$\begin{aligned} x.[b,a] =& [x.b,a] + [b,x.a] = [a,a] + [b,0] = 0+0 = 0, \\ h.[b,a] =& [h.b,a] + [b,h.a] = [-b,a] + [b,a] = -[b,a] + [b,a] = 0, \\ y.[b,a] =& [y.b,a] + [b,y.a] = [0,a] + [b,b] = 0+0 = 0. \end{aligned}$$

This completes the proof. □

For $i=4$, $[[[b,a],a],a]$ has weight 2, $[[[b,a],a],b]$ has weight 0 and $[[[b,a],b],b]$ has weight -2. Moreover, B_4^0 and B_4^2 have the same dimension. Moreover,

$$\begin{aligned} x.[[[b,a],a],b] &= [[x.b,a],a],b] + [[[b,x.a],a],b] + [[[b,a],x.a],b] + [[[b,a],a],x.b] \\ &= [[a,a],a],b] + [[[b,0],a],b] + [[[b,a],0],b] + [[[b,a],a],a] \\ &= 0+0+0+[[[b,a],a],a] = [[[b,a],a],a]. \end{aligned}$$

This tells us that $x : L_4^0 \longrightarrow L_4^2$ is injective since x is linear and kernel of x is zero. So there is no Lie invariant for degree 4.

Theorem 3.3.2. Any nonzero scalar multiple of $[[[b,a],b],[[b,a],a]]$ is a primitive Lie invariant of degree 6 for the natural representation of $sl(2)$.

Proof. For $i=6$, we know $B_6^0=\{1,3,7\}$ and $B_6^2=\{2,6\}$. We compute the actions on the basis of L_6^0 by x and obtain following equations:

$$\begin{aligned}
x.X_6^1 &= x.[[[b,a],b],[[b,a],a]] \\
&= [[[x.b,a],b],[[b,a],a]] + [[[b,x.a],b],[[b,a],a]] + [[[b,a],x.b],[[b,a],a]] \\
&\quad + [[[b,a],b],[[x.b,a],a]] + [[[b,a],b],[[b,x.a],a]] + [[[b,a],b],[[b,a],x.a]] \\
&= [[[a,a],b],[[b,a],a]] + [[[b,0],b],[[b,a],a]] + [[[b,a],a],[[b,a],a]] \\
&\quad + [[[b,a],b],[[a,a],a]] + [[[b,a],b],[[b,0],a]] + [[[b,a],b],[[b,a],0]] \\
&= 0+0+0+0+0+0 = 0 \\
x.X_6^3 &= x.[[[[b,a],a],b],[b,a]] \\
&= [[[[x.b,a],a],b],[b,a]] + [[[[b,x.a],a],b],[b,a]] + [[[[b,a],x.a],b],[b,a]] \\
&\quad + [[[[b,a],a],x.b],[b,a]] + [[[[b,a],a],b],[x.b,a]] + [[[[b,a],a],b],[b,x.a]] \\
&= [[[[a,a],a],b],[b,a]] + [[[[b,0],a],b],[b,a]] + [[[[b,a],0],b],[b,a]] \\
&\quad + [[[[b,a],a],a],[b,a]] + [[[[b,a],a],b],[a,a]] + [[[[b,a],a],b],[b,0]] \\
&= 0+0+0+[[[[b,a],a],a],[b,a]]+0+0 \\
&= [[[[b,a],a],a],[b,a]] = X_6^2 \\
x.X_6^7 &= x.[[[[[b,a],a],a],b],b] \\
&= [[[[[x.b,a],a],a],b],b] + [[[[[b,x.a],a],a],b],b] + [[[[[b,a],x.a],a],b],b] \\
&\quad + [[[[[b,a],a],x.a],b],b] + [[[[[b,a],a],a],x.b],b] + [[[[[b,a],a],a],b],x.b] \\
&= [[[[[a,a],a],a],b],b] + [[[[[b,0],a],a],b],b] + [[[[[b,a],0],a],b],b] \\
&\quad + [[[[[b,a],a],0],b],b] + [[[[[b,a],a],a],a],b] + [[[[[b,a],a],a],b],a] \\
&= 0+0+0+0+[[[[[b,a],a],a],a],b] + [[[[[b,a],a],a],b],a] \\
&= [[[[[b,a],a],a],a],b] + [[[[[b,a],a],a],a],b] + [[[[[b,a],a],a],[b,a]] \\
&\qquad\qquad\qquad\qquad \text{(by Jacobi identity)} \\
&= 2[[[[[b,a],a],a],a],b] + [[[[[b,a],a],a],[b,a]] \\
&= X_6^2 + 2X_6^6.
\end{aligned}$$

We can form a 2×3 matrix by these equations to represent $x : L_6^0 \to L_6^2$. Therefore, we obtain the following matrix equation

$$[X_6^2, X_6^6] \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = [x.X_6^1, x.X_6^3, x.X_6^7]$$

Clearly, a basis vector for the nullspace of the 2×3 matrix is $[1, 0, 0]^T$. The Lie invariant is

$$I_6 = \langle [1, 0, 0] | [X_6^1, X_6^3, X_6^7] \rangle = X_6^1 = [[[b, a], b], [[b, a], a]]$$

where $\langle * | * \rangle$ denotes the dot product. □

Example 3.3.3. We can check the result of the above theorem by following argument: Since y satisfies linearity and the Leibniz rule for derivations, we have

$$\begin{aligned} y.[[[b, a], b], [[b, a], a]] &= [[[0, a], b], [[b, a], a]] + [[[b, b], b], [[b, a], a]] + [[[b, a], 0], [[b, a], a]] \\ &\quad + [[[b, a], b], [[0, a], a]] + [[[b, a], b], [[b, b], a]] + [[[b, a], b], [[b, a], b]] \\ &= 0 + 0 + 0 + 0 + 0 + 0 = 0 \end{aligned}$$

Theorem 3.3.3. Any nonzero scalar multiple of $-X_8^5 + X_8^6$ or

$$-[[[[b, a], a], [b, a]], [[b, a], b]] + [[[[b, a], b], [b, a]], [[b, a], a]],$$

is a Lie invariant (but not primitive) of degree 8 in the natural representation of $sl(2)$.

Proof. For $i = 8$, we know

$$B_8^0 = \{2, 5, 6, 11, 12, 17, 21, 27\} \quad and \quad B_8^2 = \{1, 4, 9, 10, 16, 20, 26\}.$$

We next compute the natural actions by x on each basis vector of L_8^0.

$$\begin{aligned} x.X_8^2 &= x.[[[[b, a], b], b], [[[b, a], a], a]] \\ &= [[[[x.b, a], b], b], [[[b, a], a], a]] + [[[[b, x.a], b], b], [[[b, a], a], a]] \\ &\quad + [[[[b, a], x.b], b], [[[b, a], a], a]] + [[[[b, a], b], x.b], [[[b, a], a], a]] \end{aligned}$$

$$
\begin{aligned}
&+ [[[[b,a],b],b],[[[x.b,a],a],a]] + [[[[b,a],b],b],[[[b,x.a],a],a]]\\
&+ [[[[b,a],b],b],[[[b,a],x.a],a]] + [[[[b,a],b],b],[[[b,a],a],x.a]]\\
&= [[[[a,a],b],b],[[[b,a],a],a]] + [[[[b,0],b],b],[[[b,a],a],a]]\\
&+ [[[[b,a],a],b],[[[b,a],a],a]] + [[[[b,a],b],a],[[[b,a],a],a]]\\
&+ [[[[b,a],b],b],[[[a,a],a],a]] + [[[[b,a],b],b],[[[b,0],a],a]]\\
&+ [[[[b,a],b],b],[[[b,a],0],a]] + [[[[b,a],b],b],[[[b,a],a],0]]\\
&= [[[[b,a],a],b],[[[b,a],a],a]] + [[[[b,a],b],a],[[[b,a],a],a]]\\
&= X_8^1 + [[[[b,a],a],b],[[[b,a],a],a]] + [[[[b,a],[b,a]],[[[b,a],a],a]]\\
&= 2X_8^1
\end{aligned}
$$

$$
\begin{aligned}
x.X_8^5 &= x.[[[[b,a],a],[b,a]],[[b,a],b]]\\
&= [[[[x.b,a],a],[b,a]],[[b,a],b]] + [[[[b,x.a],a],[b,a]],[[b,a],b]]\\
&\quad + [[[[b,a],x.a],[b,a]],[[b,a],b]] + [[[[b,a],a],[x.b,a]],[[b,a],b]]\\
&\quad + [[[[b,a],a],[b,x.a]],[[b,a],b]] + [[[[b,a],a],[b,a]],[[x.b,a],b]]\\
&\quad + [[[[b,a],a],[b,a]],[[b,x.a],b]] + [[[[b,a],a],[b,a]],[[b,a],x.b]]\\
&= [[[[a,a],a],[b,a]],[[b,a],b]] + [[[[b,0],a],[b,a]],[[b,a],b]]\\
&\quad + [[[[b,a],0],[b,a]],[[b,a],b]] + [[[[b,a],a],[a,a]],[[b,a],b]]\\
&\quad + [[[[b,a],a],[b,0]],[[b,a],b]] + [[[[b,a],a],[b,a]],[[a,a],b]]\\
&\quad + [[[[b,a],a],[b,a]],[[b,0],b]] + [[[[b,a],a],[b,a]],[[b,a],a]]\\
&= + [[[[b,a],a],[b,a]],[[b,a],a]]\\
&= X_8^4
\end{aligned}
$$

$$
\begin{aligned}
x.X_8^6 &= x.[[[[b,a],b],[b,a]],[[b,a],a]]\\
&= [[[[x.b,a],b],[b,a]],[[b,a],a]] + [[[[b,x.a],b],[b,a]],[[b,a],a]]\\
&\quad + [[[[b,a],x.b],[b,a]],[[b,a],a]] + [[[[b,a],b],[x.b,a]],[[b,a],a]]\\
&\quad + [[[[b,a],b],[b,x.a]],[[b,a],a]] + [[[[b,a],b],[b,a]],[[x.b,a],a]]\\
&\quad + [[[[b,a],b],[b,a]],[[b,x.a],a]] + [[[[b,a],b],[b,a]],[[b,a],x.a]]\\
&= [[[[a,a],b],[b,a]],[[b,a],a]] + [[[[b,0],b],[b,a]],[[b,a],a]]
\end{aligned}
$$

$$
\begin{aligned}
&+ [[[[b,a],a],[b,a]],[[b,a],a]] + [[[[b,a],b],[a,a]],[[b,a],a]]\\
&+ [[[[b,a],b],[b,0]],[[b,a],a]] + [[[[b,a],b],[b,a]],[[a,a],a]]\\
&+ [[[[b,a],b],[b,a]],[[b,0],a]] + [[[[b,a],b],[b,a]],[[b,a],0]]\\
&= [[[[b,a],a],[b,a]],[[b,a],a]]\\
&= X_8^4
\end{aligned}
$$

$$
\begin{aligned}
x.X_8^{11} &= x.[[[[[b,a],a],a],b],[[b,a],b]]\\
&= [[[[[x.b,a],a],a],b],[[b,a],b]] + [[[[[b,x.a],a],a],b],[[b,a],b]]\\
&\quad + [[[[[b,a],x.a],a],b],[[b,a],b]] + [[[[[b,a],a],x.a],b],[[b,a],b]]\\
&\quad + [[[[[b,a],a],a],x.b],[[b,a],b]] + [[[[[b,a],a],a],b],[[x.b,a],b]]\\
&\quad + [[[[[b,a],a],a],b],[[b,x.a],b]] + [[[[[b,a],a],a],b],[[b,a],x.b]]\\
&= [[[[[a,a],a],a],b],[[b,a],b]] + [[[[[b,0],a],a],b],[[b,a],b]]\\
&\quad + [[[[[b,a],0],a],b],[[b,a],b]] + [[[[[b,a],a],0],b],[[b,a],b]]\\
&\quad + [[[[[b,a],a],a],a],[[b,a],b]] + [[[[[b,a],a],a],b],[[a,a],b]]\\
&\quad + [[[[[b,a],a],a],b],[[b,0],b]] + [[[[[b,a],a],a],b],[[b,a],a]]\\
&= [[[[[b,a],a],a],a],[[b,a],b]] + [[[[[b,a],a],a],b],[[b,a],a]]\\
&= X_8^9 + X_8^{10}
\end{aligned}
$$

$$
\begin{aligned}
x.X_8^{12} &= x.[[[[[b,a],a],b],b],[[b,a],a]]\\
&= [[[[[x.b,a],a],b],b],[[b,a],a]] + [[[[[b,x.a],a],b],b],[[b,a],a]]\\
&\quad + [[[[[b,a],x.a],b],b],[[b,a],a]] + [[[[[b,a],a],x.b],b],[[b,a],a]]\\
&\quad + [[[[[b,a],a],b],x.b],[[b,a],a]] + [[[[[b,a],a],b],b],[[x.b,a],a]]\\
&\quad + [[[[[b,a],a],b],b],[[b,x.a],a]] + [[[[[b,a],a],b],b],[[b,a],x.a]]\\
&= [[[[[a,a],a],b],b],[[b,a],a]] + [[[[[b,0],a],b],b],[[b,a],a]]\\
&\quad + [[[[[b,a],0],b],b],[[b,a],a]] + [[[[[b,a],a],a],b],[[b,a],a]]\\
&\quad + [[[[[b,a],a],b],a],[[b,a],a]] + [[[[[b,a],a],b],b],[[a,a],a]]\\
&\quad + [[[[[b,a],a],b],b],[[b,0],a]] + [[[[[b,a],a],b],b],[[b,a],0]]\\
&= [[[[[b,a],a],a],b],[[b,a],a]] + [[[[[b,a],a],b],a],[[b,a],a]]
\end{aligned}
$$

$$
\begin{aligned}
&= X_8^{10} + [[[[[b,a],a],a],b],[[b,a],a]] + [[[[[b,a],a],[b,a]],[[b,a],a]] \\
&= 2X_8^{10} + X_8^4 \\
x.X_8^{17} &= x.[[[[[b,a],a],b],[b,a]],[b,a]] \\
&= [[[[[x.b,a],a],b],[b,a]],[b,a]] + [[[[[b,x.a],a],b],[b,a]],[b,a]] \\
&\quad + [[[[[b,a],x.a],b],[b,a]],[b,a]] + [[[[[b,a],a],x.b],[b,a]],[b,a]] \\
&\quad + [[[[[b,a],a],b],[x.b,a]],[b,a]] + [[[[[b,a],a],b],[b,x.a]],[b,a]] \\
&\quad + [[[[[b,a],a],b],[b,a]],[x.b,a]] + [[[[[b,a],a],b],[b,a]],[b,x.a]] \\
&= [[[[[a,a],a],b],[b,a]],[b,a]] + [[[[[b,0],a],b],[b,a]],[b,a]] \\
&\quad + [[[[[b,a],0],b],[b,a]],[b,a]] + [[[[[b,a],a],a],[b,a]],[b,a]] \\
&\quad + [[[[[b,a],a],b],[a,a]],[b,a]] + [[[[[b,a],a],b],[b,0]],[b,a]] \\
&\quad + [[[[[b,a],a],b],[b,a]],[a,a]] + [[[[[b,a],a],b],[b,a]],[b,0]] \\
&= [[[[[b,a],a],a],[b,a]],[b,a]] \\
&= X_8^{16} \\
x.X_8^{21} &= x.[[[[[[b,a],a],a],b],b],[b,a]] \\
&= [[[[[[x.b,a],a],a],b],b],[b,a]] + [[[[[[b,x.a],a],a],b],b],[b,a]] \\
&\quad + [[[[[[b,a],x.a],a],b],b],[b,a]] + [[[[[[b,a],a],x.a],b],b],[b,a]] \\
&\quad + [[[[[[b,a],a],a],x.b],b],[b,a]] + [[[[[[b,a],a],a],b],x.b],[b,a]] \\
&\quad + [[[[[[b,a],a],a],b],b],[x.b,a]] + [[[[[[b,a],a],a],b],b],[b,x.a]] \\
&= [[[[[[a,a],a],a],b],b],[b,a]] + [[[[[[b,0],a],a],b],b],[b,a]] \\
&\quad + [[[[[[b,a],0],a],b],b],[b,a]] + [[[[[[b,a],a],0],b],b],[b,a]] \\
&\quad + [[[[[[b,a],a],a],a],b],[b,a]] + [[[[[[b,a],a],a],b],a],[b,a]] \\
&\quad + [[[[[[b,a],a],a],b],b],[a,a]] + [[[[[[b,a],a],a],b],b],[b,0]] \\
&= [[[[[[b,a],a],a],a],b],[b,a]] + [[[[[[b,a],a],a],b],a],[b,a]] \\
&= X_8^{20} + [[[[[[b,a],a],a],b],a],[b,a]] \\
&= X_8^{20} + [[[[[[b,a],a],a],a],b],[b,a]] + [[[[[[b,a],a],a],a],[b,a]],[b,a]] \\
&= 2X_8^{20} + X_8^{16}
\end{aligned}
$$

$$
\begin{aligned}
x.X_8^{27} &= x.[[[[[[b,a],a],a],a],b],b],b] \\
&= [[[[[[x.b,a],a],a],a],b],b],b] + [[[[[[b,x.a],a],a],a],b],b],b] \\
&\quad + [[[[[[b,a],x.a],a],a],b],b],b] + [[[[[[b,a],a],x.a],a],b],b],b] \\
&\quad + [[[[[[b,a],a],a],x.a],b],b],b] + [[[[[[b,a],a],a],a],x.b],b],b] \\
&\quad + [[[[[[b,a],a],a],a],b],x.b],b] + [[[[[[b,a],a],a],a],b],b],x.b] \\
&= [[[[[[a,a],a],a],a],b],b],b] + [[[[[[b,0],a],a],a],b],b],b] \\
&\quad + [[[[[[b,a],0],a],a],b],b],b] + [[[[[[b,a],a],0],a],b],b],b] \\
&\quad + [[[[[[b,a],a],a],0],b],b],b] + [[[[[[b,a],a],a],a],a],b],b] \\
&\quad + [[[[[[b,a],a],a],a],b],a],b] + [[[[[[b,a],a],a],a],b],b],a] \\
&= [[[[[[b,a],a],a],a],a],b],b] + [[[[[[b,a],a],a],a],b],a],b] \\
&\quad + [[[[[[b,a],a],a],a],b],b],a] \\
&= X_8^{26} + [[[[[[b,a],a],a],a],b],a],b] + [[[[[[b,a],a],a],a],b],b],a] \\
&= X_8^{26} + [[[[[[b,a],a],a],a],a],b],b] + [[[[[[b,a],a],a],a],[b,a]],b] \\
&\quad + [[[[[[b,a],a],a],a],b],a],b] + [[[[[[b,a],a],a],a],b],[b,a]] \\
&= 2X_8^{26} + X_8^{20} + [[[[[[b,a],a],a],a],b],[b,a]] \\
&\quad + [[[[[[b,a],a],a],a],[[b,a],b]] + [[[[[[b,a],a],a],a],a],b],b] \\
&\quad + [[[[[[b,a],a],a],a],[b,a]],b] \\
&= 3X_8^{26} + 2X_8^{20} + X_8^{9} \\
&\quad + [[[[[[b,a],a],a],a],b],[b,a]] + [[[[[[b,a],a],a],a],[[b,a],b]] \\
&= 3X_8^{26} + 3X_8^{20} + 2X_8^{9}
\end{aligned}
$$

Simply, we have the following equations:

$$
\begin{array}{lll}
x.X_8^2 = 2X_8^1, & x.X_8^5 = X_8^4, & x.X_8^6 = X_8^4, \\
x.X_8^{11} = X_8^9 + X_8^{10}, & x.X_8^{12} = X_8^4 + 2X_8^{10}, & x.X_8^{17} = X_8^{16}, \\
x.X_8^{21} = X_8^{16} + 2X_8^{20}, & x.X_8^{27} = 2X_8^9 + 3X_8^{20} + 3X_8^{26}. &
\end{array}
$$

We can form a 7×8 matrix A by the above equations to represent $x : L_8^0 \to L_8^2$.

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

Because the size of A is small, it is easy to see that a basis vector for the nullspace of A is $v = [0, -1, 1, 0, 0, 0, 0, 0]^T$ since the dot product of every row in A with v is the zero vector. The vector form of the basis for L_8^0 is $[X_8^2, X_8^5, X_8^6, X_8^{11}, X_8^{12}, X_8^{17}, X_8^{21}, X_8^{27}]$. The dot product of v and the vector form of the basis for L_8^0 is $-X_8^5 + X_8^6$. On the other hand, we can compute the basis vector for the nullspace by Theorem 2.7.2 which serves as the standard way to compute the nullspace of a big matrix in later sections and chapters. The Hermite normal form H of the transpose of A and the unimodular transform matrix U, where $UA^T = H$ are

$$H = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 1 & -1 & 1 \\ 0 & -1 & -1 & -4 & 2 & 3 & -3 & 2 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

By Theorem 2.8.2, the last $8 - 7 = 1$ row of U is a basis vector for the nullspace. So the Lie invariant of degree 8 is $I_8 = X_8^5 - X_8^6$, which spans the same invariant space as $-X_8^5 + X_8^6$ does. Moreover, we can show that I_8 is not primitive (§2.6). We apply standard form (Algorithm 2.4.1) to $[I_6, I_2]$ and obtain:

$$\begin{aligned} [[[[b,a],b],[[b,a],a]],[b,a]] =& [[[[b,a],b],[b,a]],[[b,a],a]] + [[[b,a],b],[[[b,a],a],[b,a]]] \\ =& [[[[b,a],b],[b,a]],[[b,a],a]] - [[[[b,a],a],[b,a]],[[b,a],b]] \\ =& -X_8^5 + X_8^6. \end{aligned}$$

This implies I_8 is not primitive. This completes the proof. □

3.4 Degree 10

All original contributions (not only the 2 generator case but also the 3 generator case) in this thesis are extensions of the following theorem and proof. The proof shows all details about how to compute Lie invariants.

Theorem 3.4.1 (Bremner, 1998,[3])**.** Any nonzero linear combination of the following 4 homogeneous Lie polynomials is a primitive Lie invariant of degree 10 in L_{10} as a $sl(2)$-module with natural action.

$$
\begin{aligned}
PI_{10}^1 =& [[[[ba]b][ba]][[[ba]a][ba]]] + [[[[[ba]a][ba]][ba]][[ba]b]] - [[[[[ba]b][ba]][ba]][[ba]a]],\\
PI_{10}^2 =& [[[[[ba]a]a]b][[[ba]b][ba]]] - 2[[[[[ba]a]b]b][[[ba]a][ba]]] - 3[[[[[ba]a]b]b][[[[ba]a]a]b]]\\
&+ [[[[[ba]b]b]b][[[[ba]a]a]a]],\\
PI_{10}^3 =& [[[[ba]b][[ba]a]][[[ba]a]b]] + [[[[[ba]a]a][[ba]b]][[ba]b]] - 2[[[[[ba]a]b][[ba]a]][[ba]b]]\\
&+ [[[[[ba]b]b][[ba]a]][[ba]a]],\\
PI_{10}^4 =& [[[[[ba]a]a[ba]][[[ba]b]b]] - 2[[[[[ba]a]b][ba]][[[ba]a]b]] + [[[[[ba]b]b][ba]][[[ba]a]a]].
\end{aligned}
$$

or simply

(1) $PI_{10}^1 = X_{10}^1 + X_{10}^{53} - X_{10}^{54}$

(2) $PI_{10}^2 = X_{10}^5 - 2X_{10}^7 - 3X_{10}^{10} + X_{10}^{13}$

(3) $PI_{10}^3 = X_{10}^{17} + X_{10}^{45} - 2X_{10}^{47} + X_{10}^{49}$

(4) $PI_{10}^4 = X_{10}^{21} - 2X_{10}^{23} + X_{10}^{25}$

where X_{10}^i denotes the ith Hall word in the Hall basis of L_{10}. See Appendix C.6 for the complete list of Hall words of degree 10.

Proof. We follow the standard way described in §3.3 to prove the theorem. First of all, we compute the weight spaces.

By definition 3.3.1, we obtain B_{10}^w for corresponding weight space L_{10}^w.

- $B_{10}^8 = \{91\}$, $|B_{10}^8| = 1$.

- $B_{10}^6 = \{28, 64, 84, 92\}$, $|B_{10}^6| = 4$.

- $B_{10}^4 = \{2, 6, 19, 29, 31, 43, 56, 65, 66, 79, 85, 93\}$, $|B_{10}^4| = 12$.

- $B_{10}^{2} = \{3, 4, 9, 16, 20, 22, 30, 32, 34, 44, 46, 52, 57, 58, 67, 68, 76, 80, 86, 94\}$, $|B_{10}^{2}| = 20$.

- $B_{10}^{0} = \{1, 5, 7, 10, 13, 17, 21, 23, 25, 33, 35, 37, 45, 47, 49, 53, 54, 59, 60, 69, 70, 77, 81, 87, 95\}$, $|B_{10}^{0}| = 25$.

- $B_{10}^{-2} = \{8, 11, 14, 18, 24, 26, 36, 38, 40, 48, 50, 55, 61, 62, 71, 72, 78, 82, 88, 96\}$, $|B_{10}^{2}| = 20$.

- $B_{10}^{-4} = \{12, 15, 27, 39, 41, 51, 63, 73, 74, 83, 89, 97\}$, $|B_{10}^{4}| = 12$.

- $B_{10}^{-6} = \{42, 75, 90, 98\}$, $|B_{10}^{6}| = 4$.

- $B_{10}^{-8} = \{99\}$, $|B_{10}^{8}| = 1$.

k	8	6	4	2	0	-2	-4	-6	-8
$dim(L_{10}^{k})$	1	4	12	20	25	20	12	4	1

Table 3.1: Dimension of Weight Spaces

The $6th$ column of table 3.1 tells the dimension of the weight space with weight 0 is 25 and the $5th$ column of the table tells the dimension of the weight space with weight 2 is 20. The difference of those two dimensions tells there are 5 highest weight modules $V(0)$ of weight 0 by Lemma 3.1.3 (§3.1). Similar arguments apply to the other pairs of consecutive columns. By Weyl's Theorem (§2.1), we obtain the decomposition of L_{10} as an $sl(2)$-module:

$$L_{10} \approx V(8) \oplus V(6)^3 \oplus V(4)^8 \oplus V(2)^8 \oplus V(0)^5,$$

where $V(k)$ denote the simple $sl(2)$-module of highest weight k, and the superscripts indicate multiplicities. There are 25 such highest weight modules. We ignore this description about the decomposition in later sections and chapters. But it should be always kept in mind.

We compute the action of x on each of the basis X_i where $i \in B_0$. (That h acts on those bases is trivial and equal to zero). Since the dimension of weight space L_{10}^{0}

is 25, we should have 25 such equations giving the action of x on the basis elements:

$$
\begin{aligned}
&x.X_{10}^{1} = 0, && x.X_{10}^{5} = X_{10}^{3} + X_{10}^{4},\\
&x.X_{10}^{7} = 2X_{10}^{4}, && x.X_{10}^{10} = -X_{10}^{4} + X_{10}^{9},\\
&x.X_{10}^{13} = 3X_{10}^{9} - X_{10}^{3}, && x.X_{10}^{17} = X_{10}^{16},\\
&x.X_{10}^{21} = 2X_{10}^{20}, && x.X_{10}^{23} = X_{10}^{20} + X_{10}^{22},\\
&x.X_{10}^{25} = 2X_{10}^{22}, && x.X_{10}^{33} = X_{10}^{30} + 2X_{10}^{32},\\
&x.X_{10}^{35} = 2X_{10}^{32} + X_{10}^{20} + X_{10}^{34}, && x.X_{10}^{37} = 3X_{10}^{34} - 2X_{10}^{16} + 3X_{10}^{22},\\
&x.X_{10}^{45} = 2X_{10}^{44} - X_{10}^{16}, && x.X_{10}^{47} = X_{10}^{44} + X_{10}^{46},\\
&x.X_{10}^{49} = 2X_{10}^{46}, && x.X_{10}^{53} = X_{10}^{52},\\
&x.X_{10}^{54} = X_{10}^{52}, && x.X_{10}^{59} = X_{10}^{57} + X_{10}^{58},\\
&x.X_{10}^{60} = 2X_{10}^{58} + X_{10}^{52}, && x.X_{10}^{69} = 2X_{10}^{67} + X_{10}^{57} + X_{10}^{68},\\
&x.X_{10}^{70} = 3X_{10}^{68} - 2X_{10}^{16} + 2X_{10}^{44} + 3X_{10}^{58}, && x.X_{10}^{77} = X_{10}^{76},\\
&x.X_{10}^{81} = 2X_{10}^{80} + X_{10}^{76}, && x.X_{10}^{87} = 3X_{10}^{86} + 2X_{10}^{3} + 2X_{10}^{57} + 3X_{10}^{80},\\
&x.X_{10}^{95} = 4X_{10}^{94} + 3X_{10}^{30} + 8X_{10}^{67} + 6X_{10}^{86}.
\end{aligned}
$$

There is an example of how we get those identities. Let's see how x acts on X_{70}:

$$
\begin{aligned}
x.X_{70} = x.&[[[[[[b,a],a],a],b],b],b][[b,a],a]]\\
&= [[[[[[x.b,a],a],a],b],b],b],[[b,a],a]] + [[[[[[b,x.a],a],a],b],b],b],[[b,a],a]]\\
&\quad + [[[[[[b,a],x.a],a],b],b],b],[[b,a],a]] + [[[[[[b,a],a],x.a],b],b],b],[[b,a],a]]\\
&\quad + [[[[[[b,a],a],a],x.b],b],b],[[b,a],a]] + [[[[[[b,a],a],a],b],x.b],b],[[b,a],a]]\\
&\quad + [[[[[[b,a],a],a],b],b],x.b],[[b,a],a]] + [[[[[[b,a],a],a],b],b],b],[[x.b,a],a]]\\
&\quad + [[[[[[b,a],a],a],b],b],b],[[b,x.a],a]] + [[[[[[b,a],a],a],b],b],b],[[b,a],x.a]]\\
&=[[[[[[a,a],a],a],b],b],b],[[b,a],a]] + [[[[[[b,0],a],a],b],b],b],[[b,a],a]]\\
&\quad + [[[[[[b,a],0],a],b],b],b],[[b,a],a]] + [[[[[[b,a],a],0],b],b],b],[[b,a],a]]\\
&\quad + [[[[[[b,a],a],a],a],b],b],[[b,a],a]] + [[[[[[b,a],a],a],b],a],b],[[b,a],a]]\\
&\quad + [[[[[[b,a],a],a],b],b],a],[[b,a],a]] + [[[[[[b,a],a],a],b],b],b],[[a,a],a]]\\
&\quad + [[[[[[b,a],a],a],b],b],b],[[b,0],a]] + [[[[[[b,a],a],a],b],b],b],[[b,a],0]]
\end{aligned}
$$

$$
\begin{aligned}
&= [[[[[[[b,a],a],a],a],b],b],[[b,a],a]] + [[[[[[[b,a],a],a],b],a],b],[[b,a],a]] \\
&\quad + [[[[[[[b,a],a],a],b],b],a],[[b,a],a]] && (3.4.1) \\
&= X_{10}^{68} + (X_{10}^{68} + [[[[[[b,a],a],a],[b,a]],b],[[b,a],a]]) + (X_{10}^{68} \\
&\quad + [[[[[[b,a],a],a],[b,a]],b],[[b,a],a]] + [[[[[[b,a],a],a],b],[b,a]],[[b,a],a]]) \\
& && (3.4.2) \\
&= 3X_{10}^{68} + 2[[[[[[b,a],a],a],[b,a]],b][[b,a],a]] + X_{10}^{58} && (3.4.3) \\
&= X_{10}^{58} + 3X_{10}^{68} + 2([[[[[[b,a],a],a],b],[b,a]],[[b,a],a]] \\
&\quad + [[[[[[b,a],a],a],[[b,a],b]][[b,a],a]]) && (3.4.4) \\
&= 3X_{10}^{58} + 3X_{10}^{68} + 2(X_{10}^{44} - X_{10}^{16}) \\
&= \; -2X_{10}^{16} + 2X_{10}^{44} + 3X_{10}^{58} + 3X_{10}^{68}
\end{aligned}
$$

We use derivation rule and action by x to obtain (3.4.1). Let us look at the second term in (3.4.1), it is *not* a Hall word. But we can apply the Jacobi identity $[[A,B],a] = [[A,a],B] + [A,[B,a]]$ to it and obtain those two terms in the first parenthesis of (3.4.2) and apply the Jacobi identity twice to the third term in (3.4.1) to get those three terms in the second parenthesis in (3.4.2). After collecting Hall words, we apply Jacobi identity $[[A,B],b] = [[A,b],B] + [A,[B,b]]$ to (3.4.2) and obtain (3.4.3) and (3.4.4). The A and B in above discussion denote some words in the free magma over $\{a,b\}$.

We form a 20×25 matrix A by these equations to represent $x : L_{10}^0 \to L_{10}^2$.

$$
A = \begin{bmatrix}
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4
\end{bmatrix}.
$$

After computing its reduced row-echelon form, and obtain the following basis for kernel:

(1) $\widetilde{I}_{10}^{1} = X_{10}^{1}$,

(2) $I_{10}^{2} = X_{10}^{5} - 2X_{10}^{7} - 3X_{10}^{10} + X_{10}^{13}$,

(3) $I_{10}^{3} = X_{10}^{17} + X_{10}^{45} - 2X_{10}^{47} + X_{10}^{49}$,

(4) $I_{10}^{4} = X_{10}^{21} - 2X_{10}^{23} + X_{10}^{25}$,

(5) $I_{10}^{5} = X_{10}^{53} - X_{10}^{54}$.

Those 5 Lie invariant span the subspace K_{10} of L_{10}.

In degree < 10, the only primitive invariants are

$$I_2 = [b, a] \quad \text{and} \quad I_6 = [[[b, a], b], [[b, a], a]].$$

The only possible way to generate Lie invariant of degree 10 by these 2 invariants is $[[I_6, I_2], I_2]$ which is equal to $2X_{10}^{1} - X_{10}^{53} + X_{10}^{54}$. It is the linear combination of $\widetilde{I}_{10}^{1}$ and I_{10}^{5} , so this linear combination is not a primitive Lie invariant. But we can define a scalar product on L_{10}^{0} by declaring Hall basis words to be an orthonormal basis. The coordinate of $[[I_6, I_2], I_2]$ is $(2, -1, 1)^T$. If $I_{10}^{1} = X_{10}^{1} + X_{10}^{53} - X_{10}^{54}$ and its coordinate is $(1, 1, -1)^T$, then $\langle (2, -1, 1)^T, (1, 1, -1)^T \rangle = 2 - 1 - 1 = 0$ which implies I_{10}^{1} is orthogonal to $[[I_6, I_2], I_2]$. Finally, we can conclude that I_{10}^{1} together with I_{10}^{2}, I_{10}^{3} and I_{10}^{4} form the basis of the space J_{10} of primitive invariants of degree 10.

As we did in degree 8 case, we can compute those Lie invariants by Theorem 2.8.2. Last 5 rows of the unimodular transform matrix U, where $UA^T = H$ and H is the Hermite normal form of the transpose of A, is

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

So we obtain Lie invariants:

(1) $I_{10}^{1} = -X_{10}^{1}$,

(2) $I_{10}^5 = X_{10}^{53} - X_{10}^{54}$,

(3) $I_{10}^4 = X_{10}^{21} - 2X_{10}^{23} + X_{10}^{25}$,

(4) $I_{10}^3 = X_{10}^{17} + X_{10}^{45} - 2X_{10}^{47} + X_{10}^{49}$,

(5) $I_{10}^2 = -X_{10}^5 + 2X_{10}^7 + 3X_{10}^{10} - X_{10}^{13}$,

We notice the Lie invariants we obtain by Theorem 2.8.2 are similar to those 5 we got before except order and sign difference. After distinguishing primitive and non-primitive Lie invariants, we complete the proof. □

The proof is basically from the main theorem appeared in [3].

3.5 Degree 12

The main theory in this section is the first original contribution in this thesis.

Theorem 3.5.1. Any nonzero linear combination of the following 4 homogeneous Lie polynomials is a primitive Lie invariant of degree 12 in the natural representation of $sl(2)$.

1. $PI1 = X_{12}^{155} - X_{12}^{157} - X_{12}^{160} + X_{12}^{162}$,

2. $PI2 = X_{12}^{44} - X_{12}^{50} - X_{12}^{55} + X_{12}^{61}$,

3. $PI3 = X_{12}^{42} - X_{12}^{47} - X_{12}^{50} - 2X_{12}^{53} + X_{12}^{55} + 2X_{12}^{58} + X_{12}^{61} + X_{12}^{64} - X_{12}^{69}$,

4. $PI4 = -X_{12}^{155} - X_{12}^{157} + X_{12}^{160} + X_{12}^{170} - X_{12}^{173} - 2X_{12}^{175} + 2X_{12}^{178} + X_{12}^{180} - X_{12}^{183}$.

The complete list of Hall words of degree 12 is in Appendix C.7.

Proof. Let's consider L_{12} generated by a, b in this section. There are 335 Hall words of degree 12. First of all, we need to find the basis B_0 of weight space L_{12}^0. Recall L_j^i denotes the weight space of weight i in L_j. After we apply the similar equation (2) in last section, we get

- $B_{12}^2 = \{1, 3, 10, 11, 13, 17, 21, 26, 38, 41, 43, 46, 49, 52, 57, 63, 73, 76, 81, 86, 89, 91, 94, 99, 114, 116, 119, 121, 124, 129, 146, 148, 154, 156, 159, 167, 169, 172, 174, 177, 190, 192, 200, 202, 204, 218, 220, 222, 234, 238, 240, 248, 250, 252, 261, 266, 267, 276, 277, 290, 291, 301, 305, 311, 319, 329\}$,

- $B_{12}^0 = \{2, 5, 14, 16, 18, 23, 27, 33, 42, 44, 47, 50, 53, 55, 58, 61, 64, 69, 74, 77, 79, 82, 90, 92, 95, 97, 100, 105, 120, 122, 125, 127, 130, 135, 147, 149, 155, 157, 160, 162, 170, 173, 175, 178, 180, 183, 191, 193, 195, 203, 205, 207, 221, 223, 225, 235, 239, 241, 243, 251, 253, 255, 262, 263, 268, 269, 278, 279, 292, 293, 302, 306, 312, 320, 330\}$,

We compute the equations giving the action of x on the basis elements of L_{12}^0. All equations are in Appendix E.1. We form a 66×75 matrix A to represent the $x : L_{12}^0 \longrightarrow L_{12}^2$. We next compute the Hermite normal form H of the transpose of the 66×75 matrix A. The last 9 rows of the unimodular transform matrix U give us 9 Lie invariants, where $UA^T = H$.

1. $I_{12}^1 = -X_{12}^{262} + X_{12}^{263}$,
2. $I_{12}^2 = 2X_{12}^{77} - X_{12}^{82} - X_{12}^{90} + X_{12}^{92} + 3X_{12}^{95} - 2X_{12}^{97} - 3X_{12}^{100} + X_{12}^{105}$,
3. $I_{12}^3 = X_{12}^{155} - X_{12}^{157} - X_{12}^{160} + X_{12}^{162}$,
4. $I_{12}^4 = X_{12}^{155} + X_{12}^{157} - X_{12}^{160} - X_{12}^{170} + X_{12}^{173} + 2X_{12}^{175} - 2X_{12}^{178} - X_{12}^{180} + X_{12}^{183}$,
5. $I_{12}^5 = X_{12}^{191} - 2X_{12}^{193} + X_{12}^{195}$,
6. $I_{12}^6 = -X_{12}^{2} + X_{12}^{239} - 2X_{12}^{241} + X_{12}^{243}$,
7. $I_{12}^7 = X_{12}^{44} - X_{12}^{50} - X_{12}^{55} + X_{12}^{61}$,
8. $I_{12}^8 = -X_{12}^{42} + X_{12}^{44} + X_{12}^{47} + 2X_{12}^{53} - 2X_{12}^{55} - 2X_{12}^{58} - X_{12}^{64} + X_{12}^{69}$,
9. $I_{12}^9 = -X_{12}^{74} + X_{12}^{79}$.

Dimension table (Appendix B) tells us that there are only 4 primitive invariants; 5 non-primitive invariants of degree 12 can be constructed by primitive invariants of lower degrees by Lie bracket. Algorithm 2.4.1 is applied here to transform any element in L_{12} to a linear combination of Hall words during the computing of non-primitive invariants.

1. $[[[I_6, I_2], I_2], I_2] = -3X_{12}^{74} + 3X_{12}^{79} - X_{12}^{262} + X_{12}^{263}$,
2. $[I_{10}^1, I_2] = X_{12}^{262} - X_{12}^{263}$,
3. $[I_{10}^2, I_2] = 2X_{12}^{77} - X_{12}^{82} - X_{12}^{90} + X_{12}^{92} + 3X_{12}^{95} - 2X_{12}^{97} - 3X_{12}^{100} + X_{12}^{105}$,
4. $[I_{10}^3, I_2] = -X_{12}^{2} + X_{12}^{157} + X_{12}^{160} + 2X_{12}^{44} - X_{12}^{162} + X_{12}^{239} - 2X_{12}^{50} - X_{12}^{55}$
$- 2X_{12}^{241} + 2X_{12}^{61} - X_{12}^{155} + X_{12}^{243}$,
5. $[I_{10}^4, I_2] = X_{12}^{191} - 2X_{12}^{193} + X_{12}^{195}$.

The coordinates of the 5 non-primitive invariants forms a 5×75 matrix with rank 5 and the coordinates of the 9 invariants in the basis of the kernel forms a 9×75 with rank 9. We add one row of the 9×75 matrix to the 5×75 matrix every time and compute the rank of that new matrix. If the rank increases by 1, the row we just added is linearly independent of the non-primitive invariants. Since $9 - 5 = 4$, if we can find 4 rows in this process, all of such rows form the basis for J_{12}. Actually, the following 4 invariants form the basis of J_{12}:

$$\begin{aligned}
PI1 &= I_{12}^3 = X_{12}^{155} - X_{12}^{157} - X_{12}^{160} + X_{12}^{162},\\
PI2 &= I_{12}^7 = X_{12}^{44} - X_{12}^{50} - X_{12}^{55} + X_{12}^{61},\\
PI3 &= I_{12}^7 - I_{12}^8 = X_{12}^{42} - X_{12}^{47} - X_{12}^{50} - 2X_{12}^{53} + X_{12}^{55} + 2X_{12}^{58} + X_{12}^{61} + X_{12}^{64} - X_{12}^{69},\\
PI4 &= I_{12}^4 = -X_{12}^{155} - X_{12}^{157} + X_{12}^{160} + X_{12}^{170} - X_{12}^{173} - 2X_{12}^{175} + 2X_{12}^{178} + X_{12}^{180} - X_{12}^{183}.
\end{aligned}$$

This completes the proof. □

Example 3.5.1. Let's check whether $PI1$ is an invariant or not.

$$\begin{aligned}
x.(X_{155} - X_{157} - X_{160} + X_{162}) &= x.X_{155} - x.X_{157} - x.X_{160} + x.X_{162}\\
&= 2X_{154} - (X_{154} + X_{156}) - (X_{154} + X_{159}) + (X_{156} + X_{159})
\end{aligned}$$

$$= 0$$

We don't give the details for the above calculation since after the first equal sign, it consists of 48 degree 12 monomials in explicit form. Direct checks are done by the MAPLE program. Similar arguments show that $PI2$, $PI3$ and $PI4$ are invariants as well by a MAPLE program. We conclude that those 4 primitive invariants form the basis for J_{12}.

3.6 Degree 14

Theorem 3.6.1. Any nonzero linear combination of I_{14}^i for $i = 1, 2, 4, 6, 7, 8, 9, 10, 12,$ $14, 17, 18, 20, 21, 22, 23, 24, 28, 29, 30, 31, 32, 33$ is a primitive Lie invariant of degree 14 in the natural representation of $sl(2)$. (The explicit invariant Lie polynomials are in the last part of the following proof.)

Proof. The integer sets B_{14}^0 and B_{14}^2 corresponding weight spaces L_{14}^0 and L_{14}^2 are:

- $B_{10}^2 = \{3, 4, 7, 17, 18, 22, 32, 33, 36, 38, 39, 43, 46, 54, 72, 78, 82, 83, 86,$ $89, 93, 94, 98, 101, 106, 114, 119, 133, 154, 156, 160, 164, 168, 176, 181, 183, 187,$ $191, 195, 200, 204, 212, 220, 224, 226, 228, 232, 235, 237, 241, 245, 249, 254, 258,$ $266, 275, 289, 291, 295, 299, 303, 311, 319, 323, 325, 327, 331, 335, 339, 347, 369,$ $373, 377, 379, 381, 385, 389, 393, 401, 425, 428, 430, 433, 438, 442, 445, 450, 465,$ $467, 470, 472, 475, 479, 482, 484, 487, 492, 496, 499, 504, 516, 532, 535, 540, 545,$ $548, 550, 553, 556, 559, 564, 570, 585, 587, 590, 593, 596, 598, 601, 604, 607, 612,$ $618, 640, 643, 648, 653, 656, 658, 661, 666, 681, 683, 686, 688, 691, 696, 723, 725,$ $728, 730, 733, 738, 761, 768, 769, 773, 778, 788, 792, 793, 797, 802, 816, 818, 820,$ $824, 826, 832, 842, 844, 847, 855, 857, 860, 862, 865, 882, 885, 887, 890, 892, 895,$ $914, 916, 924, 926, 928, 942, 944, 946, 966, 968, 970, 986, 989, 995, 998, 1001, 1010,$ $1012, 1020, 1022, 1024, 1038, 1040, 1042, 1054, 1059, 1060, 1069, 1070, 1083, 1084,$ $1101, 1102, 1114, 1118, 1124, 1132, 1142, 1154\},$

- $B_{14}^0 = \{5, 6, 8, 9, 11, 19, 20, 23, 24, 28, 34, 40, 41, 44, 47, 48, 52, 55, 56,$ $64, 84, 90, 95, 96, 99, 102, 107, 108, 112, 115, 120, 121, 129, 134, 149, 157, 161,$

$163, 165, 169, 173, 177, 184, 188, 190, 192, 196, 199, 201, 205, 209, 213, 225, 229,$
$233, 238, 242, 244, 246, 250, 253, 255, 259, 263, 267, 272, 276, 284, 292, 296, 298,$
$300, 304, 308, 312, 324, 328, 332, 334, 336, 340, 344, 348, 356, 378, 382, 386, 388,$
$390, 394, 398, 402, 410, 429, 431, 434, 436, 439, 443, 446, 448, 451, 456, 471, 473,$
$476, 483, 485, 488, 490, 493, 497, 500, 502, 505, 510, 514, 517, 522, 533, 536, 538,$
$541, 549, 551, 554, 557, 560, 562, 565, 568, 571, 576, 591, 597, 599, 602, 605, 608,$
$610, 613, 616, 619, 624, 630, 641, 644, 646, 649, 657, 659, 662, 664, 667, 672, 687,$
$689, 692, 694, 697, 702, 729, 731, 734, 736, 739, 744, 762, 763, 770, 774, 775, 779,$
$789, 794, 798, 799, 803, 808, 819, 821, 825, 827, 829, 833, 835, 843, 845, 848, 850,$
$858, 861, 863, 866, 868, 871, 888, 891, 893, 896, 898, 901, 915, 917, 919, 927, 929,$
$931, 945, 947, 949, 969, 971, 973, 987, 990, 996, 999, 1002, 1005, 1011, 1013, 1015,$
$1023, 1025, 1027, 1041, 1043, 1045, 1055, 1056, 1061, 1062, 1071, 1072, 1085,$
$1086, 1103, 1104, 1115, 1119, 1125, 1133, 1143, 1155\}.$

The size of B^0_{14} is 245, so is $\dim L^0_{14}$. We compute the natural actions on L^0_{14} by x and all equations we obtain are in Appendix E.2. We form a 212×245 matrix to represent the $x : L^0_{14} \longrightarrow L^2_{14}$. Then we compute the Hermite normal form of the transpose of the 212×245 matrix. The last 33 rows of the unimodular transform matrix provide us the 33 Lie invariants we want.

$I^1_{14} = -X^{1055}_{14} + X^{1056}_{14},$

$I^2_{14} = -X^{843}_{14} + X^{845}_{14} + X^{848}_{14} - X^{850}_{14},$

$I^3_{14} = -X^{641}_{14} + X^{646}_{14},$

$I^4_{14} = -X^{533}_{14} + X^{538}_{14},$

$I^5_{14} = X^{28}_{14},$

$I^6_{14} = -X^{551}_{14} + X^{557}_{14} + X^{562}_{14} - X^{568}_{14},$

$I^7_{14} = X^{431}_{14} - X^{436}_{14} - X^{443}_{14} + X^{448}_{14},$

$I^8_{14} = -X^{915}_{14} + 2X^{917}_{14} - X^{919}_{14},$

$$I_{14}^{9} = X_{14}^{843} + X_{14}^{845} - X_{14}^{848} - X_{14}^{858} + X_{14}^{861} + 2X_{14}^{863} - 2X_{14}^{866} - X_{14}^{868} + X_{14}^{871},$$

$$I_{14}^{10} = X_{14}^{762} - X_{14}^{763} - X_{14}^{819} + X_{14}^{821} + X_{14}^{825} - X_{14}^{829} - X_{14}^{833} + X_{14}^{835},$$

$$I_{14}^{11} = -X_{14}^{298} + X_{14}^{1011} - 2X_{14}^{1013} + X_{14}^{1015},$$

$$I_{14}^{12} =]-X_{14}^{163} + 2X_{14}^{762} - 2X_{14}^{763},$$

$$I_{14}^{13} = -X_{14}^{19} + X_{14}^{20} + X_{14}^{23} - X_{14}^{24},$$

$$I_{14}^{14} = -X_{14}^{5} - X_{14}^{6} - X_{14}^{9} + X_{14}^{11},$$

$$I_{14}^{15} = -X_{14}^{190} + X_{14}^{199},$$

$$I_{14}^{16} = +X_{14}^{292} - 2X_{14}^{300} + X_{14}^{308},$$

$$I_{14}^{17} = +X_{14}^{157} - X_{14}^{165} + X_{14}^{173},$$

$$I_{14}^{18} = +X_{14}^{5} - X_{14}^{6} - X_{14}^{8} + X_{14}^{9},$$

$$I_{14}^{19} = +X_{14}^{184} - X_{14}^{192} - X_{14}^{201} + X_{14}^{209},$$

$$I_{14}^{20} = -X_{14}^{549} + X_{14}^{554} + X_{14}^{557} + 2X_{14}^{560} - X_{14}^{562} - 2X_{14}^{565} - X_{14}^{568} - X_{14}^{571} + X_{14}^{576},$$

$$I_{14}^{21} = -X_{14}^{431} - X_{14}^{436} + X_{14}^{448} + X_{14}^{473} - 2X_{14}^{485} - X_{14}^{490} + X_{14}^{497} + 2X_{14}^{502} - X_{14}^{514},$$

$$I_{14}^{22} = -2X_{14}^{644} + X_{14}^{649} + X_{14}^{657} - X_{14}^{659} - 3X_{14}^{662} + 2X_{14}^{664} + 3X_{14}^{667} - X_{14}^{672},$$

$$\begin{aligned} I_{14}^{23} = {} & +X_{14}^{429} - X_{14}^{439} - X_{14}^{446} + X_{14}^{451} - X_{14}^{471} + X_{14}^{476} + X_{14}^{483} + X_{14}^{485} + X_{14}^{488} - X_{14}^{490} \\ & - 2X_{14}^{493} - X_{14}^{497} - 2X_{14}^{500} + X_{14}^{505} + X_{14}^{510} + X_{14}^{514} + X_{14}^{517} - X_{14}^{522}, \end{aligned}$$

$$I_{14}^{24} = -X_{14}^{184} - X_{14}^{192} + X_{14}^{201} + X_{14}^{229} - X_{14}^{238} - 2X_{14}^{246} + 2X_{14}^{255} + X_{14}^{263} - X_{14}^{272},$$

$$I_{14}^{25} = +X_{14}^{429} - X_{14}^{431} - 2X_{14}^{434} + X_{14}^{436} + X_{14}^{439} - X_{14}^{446} + X_{14}^{448} + 2X_{14}^{451} - X_{14}^{456},$$

$$I_{14}^{26} = -X_{14}^{44} + 2X_{14}^{52} + 3X_{14}^{55} - X_{14}^{64},$$

$$I_{14}^{27} = -X_{14}^{19} - X_{14}^{20} + X_{14}^{24} + X_{14}^{34} - 2X_{14}^{40} - X_{14}^{41} + X_{14}^{47} + 2X_{14}^{48} - X_{14}^{56},$$

$$\begin{aligned} I_{14}^{28} = {} & -X_{14}^{190} - 3X_{14}^{244} + 3X_{14}^{253} - X_{14}^{536} + X_{14}^{538} + 2X_{14}^{541} - 2X_{14}^{987} + X_{14}^{990} + X_{14}^{996} \\ & - 3X_{14}^{999} + 3X_{14}^{1002} - X_{14}^{1005}, \end{aligned}$$

$$
\begin{aligned}
I_{14}^{29} = {} & -X_{14}^{533} - 2X_{14}^{536} - X_{14}^{538} - 2X_{14}^{541} - X_{14}^{549} - 4X_{14}^{554} + 4X_{14}^{560} + 2X_{14}^{562} + 5X_{14}^{565} \\
& - 2X_{14}^{571} - X_{14}^{576} + X_{14}^{591} - X_{14}^{597} - X_{14}^{599} - 3X_{14}^{602} + X_{14}^{605} + 3X_{14}^{608} + 2X_{14}^{610} + 3X_{14}^{613} \\
& - 2X_{14}^{616} - 3X_{14}^{619} - X_{14}^{624} + X_{14}^{630},
\end{aligned}
$$

$$
\begin{aligned}
I_{14}^{30} = {} & 2X_{14}^{161} + X_{14}^{163} - X_{14}^{169} + X_{14}^{173} + 2X_{14}^{177} - 2X_{14}^{762} - 2X_{14}^{763} - 5X_{14}^{770} + 3X_{14}^{774} + 6X_{14}^{775} \\
& - 3X_{14}^{779} + X_{14}^{789} - 4X_{14}^{794} + 2X_{14}^{798} + 4X_{14}^{799} - 4X_{14}^{803} + X_{14}^{808},
\end{aligned}
$$

$$
\begin{aligned}
I_{14}^{31} = {} & X_{14}^{184} - 2X_{14}^{188} - X_{14}^{190} + X_{14}^{192} + 2X_{14}^{196} - X_{14}^{199} - X_{14}^{201} + X_{14}^{205} + X_{14}^{209} - X_{14}^{213} \\
& + X_{14}^{225} - X_{14}^{229} - X_{14}^{233} - 2X_{14}^{238} - 3X_{14}^{242} - 2X_{14}^{244} + 5X_{14}^{246} + 3X_{14}^{250} - 2X_{14}^{253} + 4X_{14}^{255} \\
& + 3X_{14}^{259} - 4X_{14}^{263} - 3X_{14}^{267} - X_{14}^{272} - X_{14}^{276} + X_{14}^{284},
\end{aligned}
$$

$$
\begin{aligned}
I_{14}^{32} = {} & -X_{14}^{184} + 5X_{14}^{192} + 4X_{14}^{196} - 5X_{14}^{201} - 4X_{14}^{205} + X_{14}^{229} - X_{14}^{238} - 2X_{14}^{246} + 2X_{14}^{255} + X_{14}^{263} \\
& - X_{14}^{272} + 2X_{14}^{292} - 5X_{14}^{296} - 5X_{14}^{298} + 2X_{14}^{300} + 9X_{14}^{304} + 2X_{14}^{308} - 3X_{14}^{312} + X_{14}^{324} - 3X_{14}^{328} \\
& - 4X_{14}^{332} - 4X_{14}^{334} + 9X_{14}^{336} + 6X_{14}^{340} - 5X_{14}^{344} - 4X_{14}^{348} + X_{14}^{356} - X_{14}^{1011} + 2X_{14}^{1013} - X_{14}^{1015},
\end{aligned}
$$

$$
\begin{aligned}
I_{14}^{33} = {} & 21X_{14}^{5} + 5X_{14}^{6} - 3X_{14}^{8} - 19X_{14}^{9} + 7X_{14}^{11} + 4X_{14}^{19} + 8X_{14}^{20} - 7X_{14}^{23} - 2X_{14}^{24} - 6X_{14}^{34} \\
& - 35X_{14}^{40} + 28X_{14}^{41} + 9X_{14}^{44} + 21X_{14}^{47} - 18X_{14}^{48} - 12X_{14}^{52} + 12X_{14}^{55} + 9X_{14}^{56} + 6X_{14}^{64} \\
& + 5X_{14}^{84} + 6X_{14}^{90} - 25X_{14}^{95} + 5X_{14}^{96} - 3X_{14}^{99} - 24X_{14}^{102} + 15X_{14}^{107} + 5X_{14}^{108} + 9X_{14}^{112} \\
& + 26X_{14}^{115} + 10X_{14}^{120} - 5X_{14}^{121} - 9X_{14}^{129} - 5X_{14}^{134} + X_{14}^{149}.
\end{aligned}
$$

Dimension table (Appendix B) tells us that there are 23 primitive invariants. We can construct the following non-primitive invariants of degree 14 from primitive invariants of lower degrees by Lie bracket operation. It is not hard to find out that $[I_{12}^i, [b, a]]$, where $1 \le i \le 9$, are non-primitive invariants of degree 14. Another one is $[[I_6, [b, a]], I_6]$.

1. $NPI_{14}^1 = +X_{14}^{641} - X_{14}^{646} + X_{14}^{1055} - X_{14}^{1056}$,

2. $NPI_{14}^2 = -2X_{14}^{28} + X_{14}^{641} - X_{14}^{646}$,

3. $NPI_{14}^3 = +X_{14}^{184} - X_{14}^{192} - X_{14}^{201} + X_{14}^{209} + X_{14}^{843} - X_{14}^{845} - X_{14}^{848} + X_{14}^{850}$,

4. $NPI_{14}^4 = +X_{14}^{19} - X_{14}^{20} - X_{14}^{23} + X_{14}^{24} + X_{14}^{431} - X_{14}^{436} - X_{14}^{443} + X_{14}^{448} + X_{14}^{551} - X_{14}^{557} - X_{14}^{562} + X_{14}^{568}$,

5. $NPI_{14}^5 = -X_{14}^{292} + 2X_{14}^{300} - X_{14}^{308} - X_{14}^{915} + 2X_{14}^{917} - X_{14}^{919}$,

6. $NPI^{6}_{14} = +X^{184}_{14} - X^{192}_{14} - X^{201}_{14} + X^{209}_{14} + X^{298}_{14} - 2X^{551}_{14} + 2X^{557}_{14} + 2X^{562}_{14} - 2X^{568}_{14} - X^{1011}_{14} + 2X^{1013}_{14} - X^{1015}_{14}$,

7. $NPI^{7}_{14} = -X^{19}_{14} - X^{20}_{14} + X^{24}_{14} + X^{34}_{14} - 2X^{40}_{14} - X^{41}_{14} + X^{47}_{14} + 2X^{48}_{14} - X^{56}_{14} - X^{429}_{14} + X^{431}_{14} + 2X^{434}_{14} - X^{436}_{14} - X^{439}_{14} + X^{446}_{14} - X^{448}_{14} - 2X^{451}_{14} + X^{456}_{14} + X^{549}_{14} - X^{554}_{14} - X^{557}_{14} - 2X^{560}_{14} + X^{562}_{14} + 2X^{565}_{14} + X^{568}_{14} + X^{571}_{14} - X^{576}_{14}$,

8. $NPI^{8}_{14} = -X^{184}_{14} - X^{192}_{14} + X^{201}_{14} + X^{229}_{14} - X^{238}_{14} - 2X^{246}_{14} + 2X^{255}_{14} + X^{263}_{14} - X^{272}_{14} + X^{429}_{14} - 2X^{431}_{14} - 2X^{434}_{14} + X^{439}_{14} - X^{446}_{14} + 2X^{448}_{14} + 2X^{451}_{14} - X^{456}_{14} + X^{473}_{14} - 2X^{485}_{14} - X^{490}_{14} + X^{497}_{14} + 2X^{502}_{14} - X^{514}_{14} - X^{843}_{14} - X^{845}_{14} + X^{848}_{14} + X^{858}_{14} - X^{861}_{14} - 2X^{863}_{14} + 2X^{866}_{14} + X^{868}_{14} - X^{871}_{14}$,

9. $NPI^{9}_{14} = -2X^{44}_{14} + 4X^{52}_{14} + 6X^{55}_{14} - 2X^{64}_{14} - 2X^{644}_{14} + X^{649}_{14} + X^{657}_{14} - X^{659}_{14} - 3X^{662}_{14} + 2X^{664}_{14} + 3X^{667}_{14} - X^{672}_{14}$,

10. $NPI^{10}_{14} = -X^{190}_{14} + X^{199}_{14}$.

The coordinates of those 10 non-primitive invariants form a 10×245 matrix with rank 10. We use the same method as we did in degree 12 case to find out that the primitive Lie invariants of degree 14 are I^{i}_{14} for $i = 1, 2, 4, 6, 7, 8, 9, 10, 12, 14, 17, 18, 20, 21, 22, 23, 24, 28, 29, 30, 31, 32, 33$. They form a basis of J_{14}. Correctness of those Lie invariants is checked by MAPLE. $\square$

Chapter 4

Invariants in the Adjoint Representation of $sl(2)$

In this chapter, we compute the invariant Lie polynomials in the adjoint representation of $sl(2)$. The free Lie algebra L we consider from now on is generated by 3 elements, say a, b and c. We will show that there are no Lie invariants for degree ≤ 4 and there is only one invariant in degree 5 and one in degree 6. This implies that the lowest degree non-primitive Lie invariant is of degree 11. But there exist 16104 Hall words for degree 11, so we don't compute the Lie invariants of that high degree in the present thesis. Due to this reason, all Lie invariants we compute are primitive in this chapter. The method to determine Lie invariants in this chapter is similar to the one we used in Chapter 3 except the free Lie algebra is "larger". For more details, please review the *analysis for the main problem* paragraph in §3.3.

4.1 Hall Words of 3 generators

Let $A = \{a, b, c\}$, and let the total order on A to be $a <_A b <_A c$. All definitions about Hall words of 2 generators in §2.4, (pp17-18) are equally applied to Hall words of 3 generators case. For elements in $M(A) \setminus A$, we can define the total order $<_{M(A)}$ or simply $<_M$ on $M(A)$ by making $<_M$ agree with $<_A$ on A. Recall that $M(A)$ is the free magma over set $\{a, b, c\}$. By Definition 2.4.1, we list the tables of Hall words from degree 1 to degree 4 by Tables 4.1 $-$ 4.4. The subscript of each Hall word of degree d denotes its position in the Hall basis of degree d.

The recursive computation to generate Hall words is similar to the one in §2.4

$$a_1, \quad b_2, \quad c_3.$$

Table 4.1: Hall words of degree 1 in 3 generators

$$[b,a]_1, \quad [c,a]_2, \quad [c,b]_3.$$

Table 4.2: Hall words of degree 2 in 3 generators

$$\begin{array}{llll} [[b,a],a]_1, & [[b,a],b]_2, & [[b,a],c]_3, & [[c,a],a]_4, \\ [[c,a],b]_5, & [[c,a],c]_6, & [[c,b],b]_7, & [[c,b],c]_8. \end{array}$$

Table 4.3: Hall words of degree 3 in 3 generators

$$\begin{array}{lllll} [[c,a],[b,a]]_1, & [[c,b],[b,a]]_2, & [[c,b],[c,a]]_3, & [[[b,a],a],a]_4, & [[[b,a],a],b]_5, \\ [[[b,a],a],c]_6, & [[[b,a],b],b]_7, & [[[b,a],b],c]_8, & [[[b,a],c],c]_9, & [[[c,a],a],a]_{10}, \\ [[[c,a],a],b]_{11}, & [[[c,a],a],c]_{12}, & [[[c,a],b],b]_{13}, & [[[c,a],b],c]_{14}, & [[[c,a],c],c]_{15}, \\ [[[c,b],b],b]_{16}, & [[[c,b],b],c]_{17}, & [[[c,b],c],c]_{18}. & & \end{array}$$

Table 4.4: Hall words of degree 4 in 3 generators

$$\begin{array}{lll} x.a = 0 & x.b = -2a & x.c = b \\ h.a = 2a & h.b = 0 & h.c = -2c \\ y.a = -b & y.b = 2c & y.c = 0 \end{array}$$

Table 4.5: Adjoint Action of $sl(2)$ on a, b, c

except there are 3 generators. For chapter 4 and 5, we let L denote the free Lie algebra generated by a, b and c.

4.2 Degrees 1-5

In §3.1, we introduced the matrices for the adjoint representation of $sl(2)$. Since L_1 is a 3-dimensional vector space, we identify the basis vectors in the ordered basis $\{a, b, c\}$ with their coordinates for convenience (See §3.3).

$$a = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The actions by $sl(2)$ on a, b and c are listed in Table 4.5. Moreover, $h.a = 2a$, $h.b = 0$ and $h.c = -2c$ implies that a has weight 2, b has weight 0 and c has weight -2. So we define weight subspaces of L_i^k, where i is the degree, as follows:

Definition 4.2.1. Let L_i^k be the subspace of L_i consisting of all Hall words of degree i with weight k.

Lemma 4.2.1.

$$L_i^k = \{Z \in L_i \mid h.Z = kZ\} = span\{X_i^j \mid 2(\#a(X_i^j) - \#c(X_i^j)) = k\},$$

where $\#a(X_i^j)$ and $\#c(X_i^j)$ denote the number of a in X_i^j and the number of c in X_i^j.

Proof. Recall that $h.a = 2a$, $h.b = 0$ and $h.c = -2c$. Those actions satisfy the Leibniz rule for derivations. Suppose $X_i^k = [w_1, w_2, \cdots, w_i]$ where we ignore the Lie brackets except the outermost pair and $w_n \in \{a, b, c\}$ for $n = 1, 2, \cdots, i$. So

$$h.X_i^j = [h.w_1, w_2, \cdots, w_i] + [w_1, h.w_2, \cdots, w_i] + \cdots + [w_1, w_2, \cdots, h.w_i]. \quad (4.2.1)$$

Since the Lie bracket is bilinear and $h.w_n = 2w_n$ if $w_n = a$, $h.w_n = 0$ if $w_n = b$, $h.w_n = -2w_n$ if $w_n = c$, we can move the constant, 2 if $w_n = a$ and -2 if $w_n = c$, outside of the Lie bracket; b can't produce term since it's of weight 0. After collecting all terms on the right hand side of the equation (4.2.1), we obtain

$$h.X_i^j = jX_i^j = (2\#a(X_i^j) - 2\#c(X_i^j))X_i^j.$$

We complete the proof. □

As we explained in §3.3, the Lie invariants Z are the kernel of the mapping $x : L_i^0 \longrightarrow L_i^2$. We still use B_i^k to denote the set of subscripts of the Hall words with weight k in the basis for L_i. We list the B_i^0 and B_i^2 from degree 1 to 6:

1. $B_1^0 = \{2\}, \quad B_1^2 = \{1\}.$

2. $B_2^0 = \{2\}, \quad B_2^2 = \{1\}.$

3. $B_3^0 = \{3, 5\}, \quad B_3^2 = \{2, 4\}.$

4. $B_4^0 = \{2, 8, 12, 13\}, \quad B_4^2 = \{1, 6, 7, 11\}.$

5. $B_5^0 = \{6, 8, 12, 14, 16, 19, 30, 32, 39, 41\}, \quad B_5^2 = \{3, 5, 7, 11, 13, 29, 31, 37, 38\}.$

6. $B_6^0 = \{9, 12, 14, 17, 19, 22, 31, 33, 45, 48, 50, 52, 60, 62, 65, 67, 73, 90, 93, 102, 104, 107\}, \quad B_6^2 = \{3, 6, 8, 10, 11, 16, 30, 32, 42, 44, 47, 49, 57, 59, 61, 64, 87, 89, 92, 101, 103\}.$

We see $|B_i^0| = |B_i^2|$, when $i = 1, 2, 3, 4$. There are no Lie invariants in those cases since the map x is surjective and equal dimensions imply injective as well. But for degree 5, $|B_5^0| = 10$ and $|B_5^2| = 9$ implies that there exists a Lie invariant. Similar argument suggests that there is one invariant of degree 6.

Theorem 4.2.1. Any scalar multiple of

$$I_5 = X_5^6 + 4X_5^8 + 2X_5^{12} - 2X_5^{14} - 2X_5^{16} - X_5^{19}$$

or explicitly

$$\begin{aligned} I_5 =& [[[b, a], b], [c, b]] + 4[[[b, a], c], [c, a]] + 2[[[c, a], a], [c, b]] - 2[[[c, a], b], [c, a]] \\ & - 2[[[c, a], c], [b, a]] - [[[c, b], b], [b, a]], \end{aligned}$$

is a primitive invariant of degree 5 for the adjoint representation of $sl(2)$ where L is the free Lie algebra generated by a, b, c.

Proof. We regard L_1 as a copy of the simple highest weight $sl(2)$-module with highest weight 2. The adjoint action of $sl(2)$ is listed in Table 4.5. This action extends to all of L by linearity and the Leibniz rule for derivations.

We want to determine which linear combination of the Hall words of degree 5 are Lie invariants under the adjoint action of $sl(2)$. That is, we want to find Lie polynomials $Z \in L_5$ which satisfy $D.Z = 0$ for all $D \in sl(2)$. Since L_5 is finite dimensional, it can be decomposed as a direct sum of simple highest weight modules. It suffices to impose the conditions $x.Z = 0$ and $h.Z = 0$. By Lemma 2.1.1, the invariants $Z \in L_5$ are the kernel of the mapping $x : L_5^0 \longrightarrow L_5^2$. We already know that the weight spaces L_5^k where $k = 0, 2$ has a basis consisting of the Hall words X_5^i with $i \in B_5^k$ as follows:

$$B_5^0 = \{6, 8, 12, 14, 16, 19, 30, 32, 39, 41\} \quad \text{and} \quad B_5^2 = \{3, 5, 7, 11, 13, 29, 31, 37, 38\}.$$

We next compute the adjoint action of x on each of the Hall words in the weight space with weight 0. We obtain

$$\begin{aligned}
x.X_5^6 &= -2X_5^3 - 2X_5^5, & x.X_5^8 &= X_5^5 + X_5^7, \\
x.X_5^{12} &= X_5^3 - 2X_5^{11}, & x.X_5^{14} &= X_5^5 - 2X_5^{11} + X_5^{13}, \\
x.X_5^{16} &= X_5^7 + X_5^{13}, & x.X_5^{19} &= 2X_5^7 - 4X_5^{13}, \\
x.X_5^{30} &= X_5^3 + 2X_5^{29}, & x.X_5^{32} &= -4X_5^{29} + X_5^{31}, \\
x.X_5^{39} &= X_5^{29} - 2X_5^{37} + X_5^{38}, \\
x.X_5^{41} &= 4X_5^5 - 6X_5^{13} + X_5^{31} - 6X_5^{38}.
\end{aligned}$$

From those identities, we form a 9×10 matrix A to represent $x : L_5^0 \longrightarrow L_5^2$:

$$A = \begin{bmatrix}
-2 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 4 \\
0 & 1 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & -4 & 0 & 0 & 0 & -6 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & -4 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -6
\end{bmatrix}.$$

The Hermite normal form H for the transpose of above matrix is

$$H = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 4 & 4 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 4 & 4 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 4 & 4 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 10 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

The unimodular transform matrix U, where $UA^T = H$, is

$$U = \begin{bmatrix} 0 & 2 & -2 & 2 & -4 & 1 & 3 & 1 & -2 & -1 \\ 0 & 2 & -3 & 3 & -4 & 1 & 3 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -3 & 2 & -4 & 1 & 3 & 1 & -2 & -1 \\ 0 & 1 & -3 & 3 & -3 & 1 & 3 & 1 & -2 & -1 \\ 1 & 3 & -3 & 3 & -5 & 1 & 5 & 1 & -5 & -1 \\ 1 & 3 & -3 & 3 & -5 & 1 & 5 & 2 & -2 & -1 \\ 1 & 3 & -3 & 3 & -5 & 1 & 5 & 1 & -6 & -1 \\ 0 & 2 & -2 & 2 & -4 & 1 & 2 & 1 & 0 & -1 \\ 1 & 4 & 2 & -2 & -2 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since $Rank(A)$ is 9, the last row of the unimodular transform matrix U provides us the basis for the kernel for $x : L_5^0 \longrightarrow L_5^2$.(Theorem 2.7.2):

$$\begin{bmatrix} 1 & 4 & 2 & -2 & -2 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.2.2}$$

So the basis for the Lie invariant space

$$\begin{aligned} I_5 = {} & [[[b,a],b],[c,b]] + 4[[[b,a],c],[c,a]] + 2[[[c,a],a],[c,b]] - 2[[[c,a],b],[c,a]] \\ & - 2[[[c,a],c],[b,a]] - [[[c,b],b],[b,a]], \end{aligned}$$

We complete the proof. $\square$

Example 4.2.1. The Hall words $X_5^i (1 \leq i \leq 48)$ which form a basis of L_5 are in Appendix D. It's good idea to check whether the Lie polynomial we got above is indeed an invariant of degree 5 on 3 generators. We list the adjoint action on each monomial of the invariant polynomial:

$$\begin{aligned} x.X_5^6 &= x.[[[b,a],b],[c,b]] = -2[[[b,a],a],[c,b]] - 2[[[b,a],b],[c,a]] = -2X_5^3 - 2X_5^5, \\ x.X_5^8 &= x.[[[b,a],c],[c,a]] = [[[b,a],b],[c,a]] + [[[b,a],c],[b,a]] = X_5^5 + X_5^7, \\ x.X_5^{12} &= x.[[[c,a],a],[c,b]] = [[[b,a],a],[c,b]] - 2[[[c,a],a],[c,a]] = X_5^3 - 2X_5^{11}, \\ x.X_5^{14} &= x.[[[c,a],b],[c,a]] = [[[b,a],b],[c,a]] - 2[[[c,a],a],[c,a]] + [[[c,a],b],[b,a]] \\ &= X_5^5 - 2X_5^{11} + X_5^{13}, \\ x.X_5^{16} &= x.[[[c,a],c],[b,a]] = [[[b,a],c],[b,a]] + [[[c,a],b],[b,a]] = X_5^7 + X_5^{13}, \\ x.X_5^{19} &= x.[[[c,b],b],[b,a]] = -2[[[c,a],b],[b,a]] - 2[[[c,b],a],[b,a]] \\ &= -2X_5^{13} - 2([[[c,a],b],[b,a]] - [[[b,a],c],[b,a]]) = -2X_5^{13} - 2(X_5^{13} - X_5^7) \\ &= -4X_5^{13} + 2X_5^7. \end{aligned}$$

By linearity and Leibniz rule, we obtain

$$\begin{aligned}
& x.(X_5^6 + 4X_5^8 + 2X_5^{12} - 2X_5^{14} - 2X_5^{16} - X_5^{19}) \\
= \; & x.X_5^6 + 4x.X_5^8 + 2x.X_5^{12} - 2x.X_5^{14} - 2x.X_5^{16} - x.X_5^{19} \\
= \; & -2X_5^3 - 2X_5^5 + 4X_5^5 + 4X_5^7 + 2X_5^3 - 4X_5^{11} - 2X_5^5 + 4X_5^{11} - 2X_5^{13} \\
& -2X_5^7 - 2X_5^{13} + 4X_5^{13} - 2X_5^7 = 0.
\end{aligned}$$

Therefore $x.Z = 0$ as required.

4.3 Degree 6

Theorem 4.3.1. Any scalar multiple of

$$I_6 = 4X_6^{31} - 2X_6^{33} - 4X_6^{45} - X_6^{48} + 4X_6^{52} - 8X_6^{62} - 2X_6^{65} - 4X_6^{67} - X_6^{73}$$

or explicitly

$$\begin{aligned}
I_6 = \; & 4[[[c,b],b],[b,a]] - 2[[[c,b],[b,a]],[c,a]] - 4[[[[b,a],a],c],[c,b]] \\
& - [[[[b,a],b],b],[c,b]] + 4[[[[b,a],c],c],[b,a]] - 8[[[[c,a],a],c],[c,a]] \\
& - 2[[[[c,a],b],b],[c,a]] - 4[[[[c,a],b],c],[b,a]] - [[[[c,b],b],b],[b,a]]
\end{aligned}$$

is a primitive Lie invariant of degree 6 in the adjoint representation of $sl(2)$ where L is the free Lie algebra generated by a, b, c.

Proof. We sketch the proof since most parts of the proof are similar to the proof of Theorem 4.2.1. The weight spaces L_6^k where $k = 0, 2$ have a basis consisting of the Hall words X_6^i with $i \in B_6^k$ as follows:

$$B_6^0 = \{9, 12, 14, 17, 19, 22, 31, 33, 45, 48, 50, 52, 60, 62, 65, 67, 73, 90, 93, 102, 104, 107\},$$
$$B_6^2 = \{3, 6, 8, 10, 11, 16, 30, 32, 42, 44, 47, 49, 57, 59, 61, 64, 87, 89, 92, 101, 103\}.$$

We compute the adjoint action of x on each of the Hall words in the weight space with weight 0. we obtain the following equations:

$$
\begin{aligned}
&x.X_6^9 = -X_6^3 - 2X_6^6 + X_6^8, && x.X_6^{12} = X_6^3 + X_6^8 - 2X_6^{11},\\
&x.X_6^{14} = -X_6^6 + X_6^{10} + X_6^{11}, && x.X_6^{17} = 2X_6^3 - 4X_6^8 - 2X_6^{16},\\
&x.X_6^{19} = -2X_6^6 - 4X_6^{10} + X_6^{16}, && x.X_6^{22} = -2X_6^{11} + X_6^{16},\\
&x.X_6^{31} = -2X_6^{30}, && x.X_6^{33} = -2X_6^{30} + X_6^{32},\\
&x.X_6^{45} = X_6^{42} - 2X_6^{44}, && x.X_6^{48} = -4X_6^{42} - 2X_6^{47},\\
&x.X_6^{50} = -2X_6^{44} + X_6^{47} + X_6^{49}, && x.X_6^{52} = -X_6^{32} + 2X_6^{49},\\
&x.X_6^{60} = X_6^{42} - 2X_6^{57} - 2X_6^{59}, && x.X_6^{62} = X_6^{44} + X_6^{59} + X_6^{61},\\
&x.X_6^{65} = -2X_6^{30} + X_6^{47} - 4X_6^{59} + X_6^{64}, &&\\
&x.X_6^{67} = X_6^{49} - 2X_6^{61} + X_6^{64}, && x.X_6^{73} = -6X_6^{32} + 4X_6^{49} - 6X_6^{64},\\
&x.X_6^{90} = X_6^{42} - 2X_6^{87} + 2X_6^{89}, &&\\
&x.X_6^{93} = 2X_6^3 - 2X_6^{49} - 6X_6^{89} + X_6^{92}, &&\\
&x.X_6^{102} = X_6^{57} + X_6^{87} + 2X_6^{101}, &&\\
&x.X_6^{104} = -2X_6^6 - 2X_6^{61} + X_6^{89} - 4X_6^{101} + X_6^{103}, &&\\
&x.X_6^{107} = -16X_6^8 + 6X_6^{47} - 12X_6^{64} + X_6^{92} - 8X_6^{103}. &&
\end{aligned}
$$

By those equations, we form a 21×22 matrix to represent $x : L_6^0 \longrightarrow L_6^2$ and use the Hermite normal form to obtain the following Lie invariant:

$$I_6 = 4X_6^{31} - 2X_6^{33} - 4X_6^{45} - X_6^{48} + 4X_6^{52} - 8X_6^{62} - 2X_6^{65} - 4X_6^{67} - X_6^{73}. \quad (4.3.1)$$

Equation(4.3.1) spans the space K_6. We complete the proof. $\square$

Remark 4.3.1. $[I_6, I_5]$, of degree 11, is the lowest degree non-primitive invariant Lie polynomial in the adjoint representation L of $sl(2)$.

4.4 Degree 7

There are 312 Hall words of degree 7 in 3 generators. We need to determine the weight spaces L_7^0 and L_7^2. We obtain B_7^0 and B_7^2 as follows:

$$
\begin{aligned}
B_7^0 = \{&6, 7, 11, 13, 18, 20, 40, 46, 47, 54, 55, 59, 61, 66, 68, 80, 86, 87, 91, 93, 99, 101,\\
&106, 108, 113, 122, 124, 129, 150, 155, 159, 160, 167, 171, 172, 176, 182, 187, 207,
\end{aligned}
$$

$209, 213, 215, 217, 231, 234, 236, 238, 242, 244, 253, 274, 277, 281, 294, 297, 301\}$

$$B_7^2 = \{3, 5, 10, 12, 17, 32, 38, 39, 43, 45, 51, 53, 58, 60, 65, 78, 79, 83, 85, 90, 92, 98,$$
$$100, 105, 121, 149, 153, 154, 158, 165, 166, 170, 175, 181, 201, 204, 206, 208, 212,$$
$$214, 228, 230, 233, 235, 241, 273, 276, 280, 291, 293, 296\}$$

Next, we compute the actions by x on all elements in the weight space L_7^0, we obtain the following 56 equations.

$$x.X_7^6 = X_7^3 + X_7^5, \qquad x.X_7^7 = -4X_7^5 + 2X_7^3,$$
$$x.X_7^{11} = -2X_7^3 + X_7^{10}, \qquad x.X_7^{13} = -2X_7^5 + X_7^{10} - 2X_7^{12},$$
$$x.X_7^{18} = X_7^{10} - 2X_7^{17}, \qquad x.X_7^{20} = X_7^{12} + X_7^{17},$$
$$x.X_7^{40} = -2X_7^{32} - 2X_7^{38} + X_7^{39}, \qquad x.X_7^{46} = X_7^{38} + X_7^{43} + X_7^{45},$$
$$x.X_7^{47} = X_7^{39} - 4X_7^{45} + 2X_7^{43}, \qquad x.X_7^{54} = -4X_7^{38} + X_7^{51} + X_7^{53},$$
$$x.X_7^{55} = -4X_7^{39} - 4X_7^{53} + 2X_7^{51}, \qquad x.X_7^{59} = -2X_7^{43} + X_7^{51} + X_7^{58},$$
$$x.X_7^{61} = -2X_7^{45} + X_7^{53} + X_7^{58} - 2X_7^{60},$$
$$x.X_7^{66} = 2X_7^{58} - X_7^{10} - 2X_7^{65}, \qquad x.X_7^{68} = 2X_7^{60} - X_7^{12} + X_7^{65},$$
$$x.X_7^{80} = X_7^{32} - 2X_7^{78} + X_7^{79},$$
$$x.X_7^{86} = X_7^{38} - 2X_7^{78} + X_7^{83} + X_7^{85},$$
$$x.X_7^{87} = X_7^{39} - 2X_7^{79} - 4X_7^{85} + 2X_7^{83},$$
$$x.X_7^{91} = X_7^{43} + X_7^{83} + X_7^{90},$$
$$x.X_7^{93} = X_7^{45} + X_7^{85} + X_7^{90} - 2X_7^{92},$$
$$x.X_7^{99} = X_7^{51} - 4X_7^{83} - 2X_7^3 + X_7^{98},$$
$$x.X_7^{101} = X_7^{53} - 4X_7^{85} - 2X_7^5 + X_7^{98} - 2X_7^{100},$$
$$x.X_7^{106} = X_7^{58} - 2X_7^{90} + X_7^{98} - 2X_7^{105},$$
$$x.X_7^{108} = X_7^{60} - 2X_7^{92} + X_7^{100} + X_7^{105},$$
$$x.X_7^{113} = X_7^{65} + 2X_7^{105} - X_7^{17},$$
$$x.X_7^{122} = -6X_7^{98} - 6X_7^{10} + 4X_7^{58} - 2X_7^{121},$$
$$x.X_7^{124} = -6X_7^{100} - 6X_7^{12} + 4X_7^{60} + X_7^{121},$$
$$x.X_7^{129} = -4X_7^{105} + 2X_7^{65} + X_7^{121}, \qquad x.X_7^{150} = -4X_7^{149} + 2X_7^{17},$$
$$x.X_7^{155} = -2X_7^{149} + X_7^{153} - 2X_7^{154}, \qquad x.X_7^{159} = X_7^{153} - 2X_7^{158},$$
$$x.X_7^{160} = X_7^{154} + 2X_7^{158} - X_7^3, \qquad x.X_7^{167} = X_7^{149} + X_7^{165} - 2X_7^{166},$$

$x.X_7^{171} = X_7^{153} - 2X_7^{165} - 2X_7^{170},$

$x.X_7^{172} = X_7^{154} - 2X_7^{166} + 2X_7^{170} - X_7^{5},$

$x.X_7^{176} = X_7^{158} + X_7^{170} + X_7^{175},$ $\quad x.X_7^{182} = -4X_7^{170} + 2X_7^{158} + X_7^{181},$

$x.X_7^{187} = -2X_7^{175} + X_7^{181},$ $\quad x.X_7^{207} = -2X_7^{201} + X_7^{204} - 2X_7^{206},$

$x.X_7^{209} = 2X_7^{206} - X_7^{17} + X_7^{149} + X_7^{208},$

$x.X_7^{213} = -6X_7^{204} - 2X_7^{153} - 2X_7^{212},$ $\quad x.X_7^{215} = -4X_7^{206} + X_7^{212} + X_7^{214},$

$x.X_7^{217} = -2X_7^{208} + 2X_7^{214} - X_7^{10} + X_7^{153},$

$x.X_7^{231} = X_7^{201} + X_7^{228} - 2X_7^{230},$

$x.X_7^{234} = X_7^{204} - 4X_7^{228} - 2X_7^{165} - 2X_7^{233},$

$x.X_7^{236} = X_7^{206} - 2X_7^{230} + X_7^{233} + X_7^{235},$

$x.X_7^{238} = X_7^{208} + 2X_7^{235} - X_7^{12} + X_7^{165},$

$x.X_7^{242} = X_7^{212} - 6X_7^{233} + 4X_7^{154} - 6X_7^{170} + X_7^{241},$

$x.X_7^{244} = X_7^{214} - 4X_7^{235} - 2X_7^{3} + 2X_7^{158} - 2X_7^{175} + X_7^{241},$

$x.X_7^{253} = -8X_7^{241} - 16X_7^{10} + 16X_7^{153} - 12X_7^{181} + 6X_7^{214},$

$x.X_7^{274} = 3X_7^{273} + 2X_7^{32} + 3X_7^{201},$

$x.X_7^{277} = -4X_7^{273} - 4X_7^{43} + 2X_7^{65} - 2X_7^{208} + 2X_7^{276} + X_7^{204},$

$x.X_7^{281} = -8X_7^{276} - 6X_7^{51} - 6X_7^{214} + X_7^{280},$

$x.X_7^{294} = X_7^{273} - 2X_7^{291} + 2X_7^{293} + X_7^{228},$

$x.X_7^{297} = X_7^{276} - 6X_7^{293} + 4X_7^{60} - 4X_7^{90} - 6X_7^{83} - 6X_7^{235} + X_7^{296},$

$x.X_7^{301} = X_7^{280} - 10X_7^{296} + 30X_7^{53} - 40X_7^{98} + 8X_7^{212} - 20X_7^{241}.$

We form a 51×56 matrix to represent $x : L_7^0 \longrightarrow L_7^2$. The last $56 - 51 = 5$ rows of the unimodular transform matrix of the Hermite normal form for the transpose of 20×21 matrix provide the following result:

Theorem 4.4.1. Any nonzero linear combination of the following 5 homogeneous Lie polynomials is a primitive invariant of degree 7 for the adjoint representation of $sl(2)$ where L is the free Lie algebra generated by a, b, c:

$$I_7^1 = -X_7^{7} - X_7^{11} + X_7^{13} + 2X_7^{20} - X_7^{150} + X_7^{155} - 2X_7^{167} - X_7^{171} + 2X_7^{172} - 2X_7^{176} + X_7^{182} - X_7^{187},$$

$$I_7^2 = 2X_7^{6} + X_7^{7} + 2X_7^{11} - X_7^{13} - X_7^{18} - 2X_7^{20},$$

$$
\begin{aligned}
I_7^3 = {} & 2X_7^6 - 2X_7^7 + 3X_7^{13} - X_7^{18} - X_7^{40} + 2X_7^{47} + X_7^{54} - 3X_7^{61} + 2X_7^{66} - 2X_7^{80} + 2X_7^{86} \\
& - X_7^{87} - 4X_7^{91} + 2X_7^{93} - X_7^{99} + 2X_7^{101} - X_7^{106} - 2X_7^{108} + 2X_7^{113} - X_7^{124} + X_7^{129}, \\
I_7^4 = {} & - 8X_7^6 + X_7^7 - X_7^{11} - 5X_7^{13} - 2X_7^{20} + 2X_7^{40} + 8X_7^{46} + X_7^{55} + 6X_7^{61} + 8X_7^{68} + 4X_7^{80} \\
& - 4X_7^{86} + 2X_7^{87} - 8X_7^{91} + 4X_7^{93} - 2X_7^{99} - 2X_7^{101} - 2X_7^{106} - 4X_7^{108} - 4X_7^{113} - X_7^{122} \\
& - 2X_7^{129} - X_7^{150} + X_7^{155} - 2X_7^{167} - X_7^{171} + 2X_7^{172} - 2X_7^{176} + X_7^{182} - X_7^{187}, \\
I_7^5 = {} & 2X_7^7 - 8X_7^{11} + 2X_7^{13} - 6X_7^{18} - 4X_7^{20} + 4X_7^{150} + 6X_7^{155} + 8X_7^{159} + 32X_7^{160} + 12X_7^{167} \\
& - 12X_7^{172} - 20X_7^{176} - 8X_7^{182} - 4X_7^{187} + 4X_7^{207} + 16X_7^{209} + X_7^{213} + 4X_7^{215} + 4X_7^{217} \\
& + 8X_7^{231} + 2X_7^{234} - 8X_7^{236} - 8X_7^{238} - 2X_7^{242} - 6X_7^{244} - X_7^{253}.
\end{aligned}
$$

Complete list of Hall words of degree 7 is in Appendix D.3.

4.5 Degree 8

We need to determine the Lie invariants of degree 8 in this section. As usual, we compute B_8^0 and B_8^2.

$$
\begin{aligned}
B_8^0 = \{ & 2, 13, 18, 23, 29, 34, 35, 48, 54, 57, 63, 68, 74, 78, 79, 84, 85, 89, 96, 101, 106, 111, \\
& 112, 116, 125, 130, 140, 169, 175, 176, 185, 191, 192, 196, 198, 207, 208, 212, 214, 219, \\
& 221, 233, 239, 240, 244, 246, 255, 256, 260, 262, 267, 269, 276, 278, 283, 285, 290, 300, \\
& 302, 307, 309, 314, 323, 325, 330, 369, 377, 383, 384, 388, 390, 399, 400, 404, 406, 411, \\
& 413, 418, 441, 447, 448, 455, 456, 460, 462, 467, 469, 476, 478, 483, 485, 490, 507, 509, \\
& 514, 542, 546, 547, 564, 569, 575, 579, 580, 584, 594, 599, 603, 604, 609, 610, 614, 619, \\
& 626, 631, 660, 666, 668, 670, 675, 677, 679, 702, 704, 708, 710, 712, 719, 721, 733, 761, \\
& 765, 770, 785, 788, 792, 797\},
\end{aligned}
$$

$$
\begin{aligned}
B_8^2 = \{ & 1, 9, 12, 17, 22, 27, 28, 33, 39, 45, 47, 53, 56, 61, 62, 66, 67, 72, 73, 77, 83, 88, 95, \\
& 110, 115, 124, 161, 167, 168, 172, 174, 183, 184, 188, 190, 195, 197, 204, 206, 211, 213, \\
& 218, 231, 232, 236, 238, 243, 245, 252, 254, 259, 261, 266, 275, 277, 282, 299, 301, 306, \\
& 322, 361, 367, 368, 375, 376, 380, 382, 387, 389, 396, 398, 403, 405, 410, 433, 439, 440, \\
& 444, 446, 452, 454, 459, 461, 466, 475, 477, 482, 506, 540, 541, 545, 558, 563, 567, 568,
\end{aligned}
$$

573, 574, 578, 583, 593, 597, 598, 602, 608, 613, 625, 657, 659, 663, 665, 667, 674, 676, 696, 699, 701, 703, 707, 709, 718, 757, 760, 764, 769, 784, 787, 791}.

We compute the actions by x on all basis vectors of the weight space L_8^0 and obtain the 136 equations which are listed in Appendix E.3. By coefficients of those equations, we form a 127×136 matrix to represent the map $x : L_8^0 \longrightarrow L_8^2$. From this matrix, we obtain the following result:

Theorem 4.5.1. Any nonzero linear combination of the following 9 homogeneous Lie polynomials is a primitive invariant of degree 8 for the adjoint representation of $sl(2)$ where L is the free Lie algebra generated by a, b, c:

$$\begin{aligned}
I_8^1 = {} & X_8^{169} - X_8^{175} - X_8^{191} - X_8^{196} + X_8^{207} + X_8^{208} + X_8^{214} + X_8^{219} + X_8^{221} - X_8^{233} \\
& - 2X_8^{239} - X_8^{240} - X_8^{244} + X_8^{246} - X_8^{256} - X_8^{260} - X_8^{269} - X_8^{276} + X_8^{283} + 2X_8^{285} \\
& + X_8^{290} - X_8^{300} + X_8^{302} + X_8^{309} + X_8^{325} - X_8^{330}, \\
I_8^2 = {} & - 4X_8^{2} - 4X_8^{13} - X_8^{18} + 4X_8^{29} + 4X_8^{57} + X_8^{68} - 4X_8^{79} - X_8^{106}, \\
I_8^3 = {} & X_8^{169} - X_8^{175} + X_8^{185} + X_8^{191} + X_8^{207} - X_8^{208} - 3X_8^{214} - X_8^{219} + X_8^{221} + X_8^{233} \\
& + 2X_8^{239} - X_8^{240} + X_8^{244} + X_8^{246} - X_8^{260} + 2X_8^{262} + X_8^{267} - X_8^{269} + X_8^{276} - 2X_8^{278} \\
& + X_8^{283} - 2X_8^{285} - X_8^{290} - X_8^{309} - X_8^{314} + X_8^{325} - X_8^{330}, \\
I_8^4 = {} & - X_8^{169} - X_8^{175} - X_8^{176} + X_8^{191} - X_8^{196} + X_8^{198} + X_8^{207} - X_8^{214} + X_8^{221} + X_8^{233} \\
& - 2X_8^{239} - X_8^{240} + X_8^{244} - X_8^{246} - 2X_8^{255} - 3X_8^{260} + 2X_8^{262} + X_8^{269} + X_8^{276} \\
& + X_8^{283} + 2X_8^{285} - X_8^{290} - X_8^{300} - X_8^{309} + X_8^{323} + X_8^{325} + X_8^{330}, \\
I_8^5 = {} & - 2X_8^{169} + 2X_8^{175} - X_8^{192} + X_8^{198} + 2X_8^{207} - 2X_8^{214} + 2X_8^{221} + 2X_8^{233} - 2X_8^{244} \\
& - 2X_8^{255} + X_8^{256} + 2X_8^{260} - X_8^{267} - 2X_8^{276} + 2X_8^{278} - 2X_8^{290} - X_8^{302} + X_8^{307} \\
& - 2X_8^{325} + 2X_8^{330}, \\
I_8^6 = {} & 2X_8^{2} + X_8^{18} - X_8^{23} - 4X_8^{29} + 2X_8^{48} - 2X_8^{57} + 2X_8^{79} + 4X_8^{542} + 2X_8^{546} - 2X_8^{547} \\
& - X_8^{564} - 4X_8^{569} + X_8^{575} - 2X_8^{579} + 4X_8^{580} - 4X_8^{584} - 2X_8^{594} + 4X_8^{599} + X_8^{609} \\
& - 2X_8^{610} + 2X_8^{619} - X_8^{626} + X_8^{631}, \\
I_8^7 = {} & - 4X_8^{169} + 2X_8^{175} + X_8^{185} + 2X_8^{191} - 2X_8^{192} - 5X_8^{196} + 4X_8^{198} + 6X_8^{207} - 16X_8^{212} \\
& + 8X_8^{214} - 2X_8^{221} + 4X_8^{233} - 4X_8^{244} + 2X_8^{246} - 4X_8^{255} + X_8^{256} + 4X_8^{260} - 2X_8^{262}
\end{aligned}$$

$$+ X_8^{267} - 2X_8^{269} + 8X_8^{276} - 10X_8^{278} + 4X_8^{283} + X_8^{300} - 2X_8^{302} + X_8^{307} + 2X_8^{309}$$
$$- 2X_8^{314} + 6X_8^{325} - 4X_8^{330} - 4X_8^{369} - X_8^{377} + 4X_8^{383} - 8X_8^{388} + 4X_8^{390} + X_8^{399}$$
$$- 2X_8^{404} + X_8^{406} - 4X_8^{413} + 4X_8^{418} + 4X_8^{448} + X_8^{456} + 4X_8^{460} - 8X_8^{462} + 4X_8^{467}$$
$$+ X_8^{476} - 2X_8^{478} + X_8^{483} + 4X_8^{485} - 4X_8^{490} + X_8^{509} - X_8^{514},$$

$$I_8^8 = - 12X_8^{169} + 2X_8^{175} - 2X_8^{176} + 6X_8^{191} - 3X_8^{192} + 4X_8^{196} + X_8^{198} + 2X_8^{207} - 2X_8^{208}$$
$$+ 16X_8^{212} + 4X_8^{214} + 6X_8^{219} - 6X_8^{221} + 6X_8^{233} - 8X_8^{239} + 2X_8^{240} + 2X_8^{244} - 4X_8^{246}$$
$$- 6X_8^{255} + 3X_8^{256} - 4X_8^{260} - 4X_8^{262} + X_8^{267} - 4X_8^{269} - 10X_8^{276} - 10X_8^{278} + 2X_8^{283}$$
$$- 8X_8^{285} + 6X_8^{290} - 2X_8^{300} - 2X_8^{302} - 2X_8^{307} + 2X_8^{309} - 3X_8^{314} - 6X_8^{323} + 6X_8^{325}$$
$$- 12X_8^{330} - 12X_8^{369} - 3X_8^{377} + 4X_8^{383} - 4X_8^{384} + 8X_8^{388} - 4X_8^{390} + X_8^{399} - X_8^{400}$$
$$+ 2X_8^{404} - X_8^{406} + 4X_8^{411} - 4X_8^{413} + 12X_8^{418} - 16X_8^{447} + 4X_8^{448} - 4X_8^{455} + X_8^{456}$$
$$- 4X_8^{460} - 16X_8^{462} + 4X_8^{467} - 16X_8^{469} - X_8^{476} - 4X_8^{478} - 3X_8^{483} - 12X_8^{490} - X_8^{507}$$
$$+ X_8^{509} - 3X_8^{514},$$

$$I_8^9 = - 12X_8^{2} - 12X_8^{13} - 7X_8^{18} - 24X_8^{23} - 36X_8^{29} + 8X_8^{48} + 12X_8^{57} + 9X_8^{68} + 12X_8^{79}$$
$$+ X_8^{106} - 11X_8^{169} + 23X_8^{175} - 22X_8^{185} - 17X_8^{191} - 8X_8^{192} - 3X_8^{196} + 10X_8^{198}$$
$$- 63X_8^{207} - 7X_8^{208} + 32X_8^{212} - 19X_8^{214} + 17X_8^{219} + 33X_8^{221} - 5X_8^{233} - 10X_8^{239}$$
$$- 5X_8^{240} - 29X_8^{244} + 5X_8^{246} + 4X_8^{255} - 5X_8^{256} + 3X_8^{260} - 12X_8^{267} - 13X_8^{269}$$
$$+ 43X_8^{276} - 8X_8^{278} + 21X_8^{283} - 6X_8^{285} - 19X_8^{290} + X_8^{300} + 7X_8^{302} + 4X_8^{307} + 5X_8^{309}$$
$$- 4X_8^{314} + 16X_8^{323} + 9X_8^{325} - 5X_8^{330} - 8X_8^{369} - 2X_8^{377} - 4X_8^{384} + 16X_8^{388} - 8X_8^{390}$$
$$- X_8^{400} + 4X_8^{404} - 2X_8^{406} + 4X_8^{411} + 8X_8^{418} - 16X_8^{447} - 4X_8^{455} - 8X_8^{460} - 8X_8^{462}$$
$$- 16X_8^{469} - 2X_8^{476} - 2X_8^{478} - 4X_8^{483} - 4X_8^{485} - 8X_8^{490} - X_8^{507} - 2X_8^{514} - 24X_8^{542}$$
$$+ 12X_8^{547} + 12X_8^{564} + 48X_8^{569} - 16X_8^{579} - 24X_8^{580} - 48X_8^{584} + 8X_8^{594} - 16X_8^{603}$$
$$+ 48X_8^{604} + 12X_8^{610} + 48X_8^{614} + 24X_8^{619} + 8X_8^{626} + 12X_8^{631} + 16X_8^{660} + 8X_8^{666}$$
$$- 16X_8^{670} + X_8^{675} - 8X_8^{679} + 32X_8^{704} + 16X_8^{710} + 16X_8^{712} + 2X_8^{719} + 8X_8^{721} + X_8^{733}.$$

Complete list of Hall words of degree 8 is in Appendix D.4.

Remark 4.5.1. Following this pattern, we can determine the Lie invariants of degree n for $n > 8$ in adjoint representation of $sl(2)$. Moreover, from degree 11, we begin to distinguish primitive and non-primitive Lie invariants as we did in last chapter. Since Theorem 2.5.3 fails in the adjoint representation, the dimension table for 3 generators in Appendix B of Lie invariants does *not* apply to this case.

Chapter 5

Invariants in the Natural Representation of $sl(3)$

In this chapter, we want to find invariant Lie polynomials in the natural representation of $sl(3)$, where L is a free Lie algebra generated by a, b, c. Since $sl(3)$ is a 8-dimensional algebra and L is generated by 3 elements, there are 24 elements in the action table. So we must decide what elements of $sl(3)$ play the role of h in $sl(2)$ and what elements $sl(3)$ play the role of x in $sl(2)$. A number of the basic constructions in $sl(3)$ needs to be modified from $sl(2)$.

5.1 Representation of sl(3)

$$x_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad x_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad x_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$h_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad h_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$y_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad y_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad y_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Table 5.1: Standard Basis of $sl(3)$

$x_1.a = 0$	$x_1.b = a$	$x_1.c = 0$	$x_2.a = 0$	$x_2.b = 0$	$x_2.c = b$
$x_3.a = 0$	$x_3.b = a$	$x_3.c = a$	$h_1.a = a$	$h_1.b = -b$	$h_1.c = 0$
$h_2.a = 0$	$h_2.b = b$	$h_2.c = -c$	$y_1.a = b$	$y_1.b = 0$	$y_1.c = 0$
$y_2.a = 0$	$y_2.b = c$	$y_2.c = 0$	$y_3.a = c$	$y_3.b = 0$	$y_3.c = 0$

Table 5.2: Natural Action of $sl(3)$ on a, b, c

Recall Defintion 2.1.5. If $\dim V = m + 1$, then $\dim sl(m + 1) = (m + 1)^2 - 1$. When $m = 2$, we have $\dim sl(3) = (2+1)^2 - 1 = 8$. By the standard way to generate the basis for $sl(m + 1)$ in §2.1, we obtain Table 5.1 which lists the ordered basis of $sl(3)$. If we write a, b, c for the basis of 3-dimensional space, we obtain the Table 5.2. The computation to generate Table 5.2 is very similar to the action computation in §3.3.

In the representation of $sl(3)$, we have to replace the single element $h \in sl(2)$ by a subspace $span\{h_1, h_2\} \in sl(3)$ which means the weight $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2$ where λ_1 is the weight for h_1 and λ_2 is the weight for h_2. To be precise, we have $\lambda = (\lambda_1, \lambda_2)$ where $h_1.v = \lambda_1 v$ and $h_2.v = \lambda_2 v$. Obviously, λ depends on $span\{h_1, h_2\}$. We conclude the following definition.

Definition 5.1.1. A vector v in L is an **eigenvector** (a weight vector in previous chapters) for $span\{h_1, h_2\}$ if

$$h.v = \lambda v$$

for every $h \in span\{h_1, h_2\}$, where λ is the eigenvalue or the weight of v.

In fact, any finite dimensional representation V of $sl(3)$ has a decomposition

$$V = \bigoplus V_\lambda \tag{5.1.1}$$

where V_λ is a weight space (eigenspace) with weight (eigenvalue) λ. We next determine what elements in $sl(3)$ play the roles of x and y in $sl(2)$. The key is to look at the commutation relations

$$[h, x] = 2x \quad \text{and} \quad [h, y] = -2y$$

in $sl(2)$. The better way to interpret these is to say that x and y are weight vectors for the adjoint action of $h \in sl(2)$. We can use the analogue for $sl(3)$. In other words, we apply (5.1.1) to the adjoint representation of $sl(3)$ to obtain the decomposition

$$sl(3) = span\{h_1, h_2\} \oplus (\oplus g_\lambda)$$

where λ is a weight and $h \in span\{h_1, h_2\}$ acts on each weight space g_λ by scalar multiplication, i.e., for any $h \in span\{h_1, h_2\}$ and $x \in g_\lambda$

$$[h, x] = ad(h)(x) = \lambda(h) \cdot x \tag{5.1.2}$$

where the λ depends on h. More details can be found on ([6],pp162-164).

$[h_1, x_1] = 2x_1$	$[h_2, x_1] = -x_1$	$[h_1.x_2] = -x_2$	$[h_2, x_2] = 2x_2$
$[h_1, x_3] = x_3$	$[h_2, x_3] = x_3$	$[h_1.y_1] = -2y_1$	$[h_2, y_1] = y_1$
$[h_1, y_2] = y_2$	$[h_2, y_2] = -2y_2$	$[h_1.y_3] = -y_3$	$[h_2, y_3] = -y_3$

Table 5.3: Adjoint Action of $span\{h_1, h_2\}$ on $x_1, x_2, x_3, y_1, y_2, y_3$

In our present circumstances, we consider natural representation of $sl(3)$ where L is a free Lie algebra generated by a, b, c. We apply (5.1.2) to compute the weight of x_1, x_2 and x_3 under the adjoint action of $span\{h_1, h_2\}$ on $sl(3)$ to get table 5.3. By Table 5.3, we notice that under the adjoint action of $span\{h_1, h_2\}$, x_1 is of weight $(2, -1)$, x_2 is of weight $(-1, 2)$, x_3 is of weight $(1, 1)$, y_1 is of weight $(-2, 1)$, y_2 is of weight $(1, -2)$ and y_3 is of weight $(-1, -1)$. It is not surprising to see that the weights of x_i and y_i $(i = 1, 2, 3)$ have the same absolute value of each component but opposite sign. This is similar to the relations between the weights of x and y in $sl(2)$.

Lemma 5.1.1. For any $z \in g_\alpha$, $v \in V_\lambda$ and $h \in span\{h_1, h_2\}$, we have

$$h \cdot (z \cdot v) = (\alpha(h) + \lambda(h)) z \cdot v.$$

Proof. $h \cdot (z \cdot v) = z \cdot h \cdot v + [h, z](v) = z \cdot (\lambda(h)v) + (\alpha(h)z) \cdot v = (\alpha(h) + \lambda(h))(z \cdot v)$ □

That is, if v is an element in $span\{h_1, h_2\}$ with weight $(0, 0)$, then $x_1 \cdot v$ is an element in $span\{h_1, h_2\}$ with weight $(2, -1)$ and $x_2 \cdot v$ is an element in $span\{h_1, h_2\}$

with weight $(-1,2)$. To be precise, we have this because the root system of $sl(3)$ is equal to the weight system of the adjoint representation of $sl(3)$. More information can be found in [9],(§12).

The terminology "highest weight" and "highest weight module" in $sl(2)$ applies equally well to representation of $sl(3)$. We have enough background knowledge for our following research.

5.2 Degrees 1-8

In the free Lie algebra on 3 generators, Wever (1949) observed that there is no invariant Lie polynomials of degree 3 and 6 for the natural representation of $sl(3)$. He also proved that there are 4 Lie invariants of degree 9. In this section, we compute the natural actions on Hall words of degree 1 to degree 8 to verify that there is no Lie invariant of those degrees.

We notice that a weight in the representation of $sl(3)$ is in $\mathbb{Z}^2$. Weight addition and substraction are defined to be vector addition and substraction. A question arises: In the natural representation of $sl(3)$, how can we define the weight spaces? From Table 5.2, we know $h_1.a = a$ and $h_2.a = 0$ which implies the weight of a is $(1,0)$. Same arguments for b and c, we obtain the weight of b is $(-1,1)$ and the weight of c is $(0,-1)$. Since the action satisfies linearity and the Leibniz rule, the weight of a Hall word X_i^j(i is the degree of the words and j is its position in the ordered basis of L_i) of the free Lie algebra is defined to be

$$(\#a(X_i^j))(1,0) + (\#b(X_i^j))(-1,1) + (\#c(X_i^j))(0,-1). \tag{5.2.1}$$

Simplifying (5.2.1), we obtain

$$(\#a(X_i^j) - \#b(X_i^j), \#b(X_i^j) - \#c(X_i^j))$$

where $\#x(X_i^j)(x = a,b,c)$ is the number of x in a Hall word X_i^j. i.e., Hall word $[[[c,a],a],[[b,a],c]]$ is of weight $(2,-1)$ since there are 3 a's, 1 b and 2 c's which implies the weight is $(3-1, 1-2) = (2,-1)$. All Hall words generated by 3 elements from degree 5 to 9 are listed in Appendix D.

In the natural representation of $sl(2)$, Lie invariants of degree i are the kernel of the mapping $x : L_i^0 \longrightarrow L_i^2$. In the natural representation of $sl(3)$, to find Lie invariants of degree i is to find elements $Z \in L_i$ which satisfy $D \cdot Z = 0$ for all $D \in sl(3)$. But L_i is finite dimensional, it decomposes as a direct sum of simple highest weight modules, and it suffices to impose the condition $X \cdot Z = 0$ ($X \in span\{x_1, x_2, x_3\}$) and $H \cdot Z = 0$ ($H \in span\{h_1, h_2\}$). Moreover, we have $[x_1, x_2] = x_3$ and so $x_1 \cdot v = 0$ and $x_2 \cdot v = 0$ imply $x_3 \cdot v = 0$ for some Hall word v. We conclude that the Lie invariants are the intersection of the kernel $x_1 : L_i^{(0,0)} \longrightarrow L_i^{(2,-1)}$ and the kernel $x_2 : L_i^{(0,0)} \longrightarrow L_i^{(-1,2)}$. In order to compute the explicit form of these invariants, we use the matrix C_1 to represent the linear map x_1 and the matrix C_2 to represent the linear map x_2. We merge C_1 and C_2 vertically (C_1 on top of C_2) to get a new matrix C. If $v \in nullspace(C)$, then $Cv = 0$ implies $C_1 v = 0$ and $C_2 v = 0$. So $nullspace(C)$ equals to the intersection of the kernels of x_1 and x_2. This idea serves as the standard way to compute Lie invariants in this chapter.

Lemma 5.2.1. In natural representation of $sl(3)$, L_i is the subspace of L spanned by Hall words of degree i. The weight space $L_i^{(w_1,w_2)}$ is defined to be

$$\begin{aligned} L_i^{(a,b)} &= \{Z \in L_i \mid h_1 \cdot Z = w_1 Z \textit{ and } h_2 \cdot Z = w_2 Z\} \\ &= span\{X_i^j \mid (\#a(X_i^j) - \#b(X_i^j), \#b(X_i^j) - \#c(X_i^j)) = (w_1, w_2)\} \end{aligned}$$

In degree 1, there are 3 Hall words in the ordered basis which are a, b, c. There are no Lie invariants since all weights of a, b, c are not equal to $(0, 0)$.

In degree 2, Hall words are $[b, a], [c, a], [c, b]$. There are no Lie invariants since all weights of those Hall words are not equal to $(0, 0)$.

In degree 3, $[[b, a], c]$ and $[[c, a], b]$ are of weight $(0, 0)$. $[[c, a], a]$ has weight $(2, -1)$ and $[[b, a], b]]$ has weight $(-1, 2)$.

1. $x_1 \cdot [[b, a], c] = 0$ and $x_1 \cdot [[c, a], b] = [[c, a], a]$ form a matrix 1×2 matrix $[0, 1]$ to represent $x_1 : L_3^{(0,0)} \longrightarrow L_3^{(2,-1)}$.

2. $x_2 \cdot [[b, a], c] = [[b, a], b]$ and $x_2 \cdot [[c, a], b] = [[b, a], b]$ form a matrix 1×2 matrix $[1, 1]$ to represent $x_2 : L_3^{(0,0)} \longrightarrow L_3^{(-1,2)}$.

We combine those 2 matrices vertically and obtain a 2×2 matrix

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Obviously, $Rank(C) = 2$ and Rank-Nullity Theorem imply $kernel(C) = \varnothing$. So there are no invariants in this case since the intersection of mapping x_1 and x_2 is empty. This result agrees with Wever's observation for $d = q = 3$.

In degree 4, there are no Lie invariants in degree 4 since no Hall word of degree 4 has weight $(0, 0)$.

In degree 5, there are no Lie invariants in degree 5 since no Hall word of degree 5 has weight $(0, 0)$.

In degree 6,

$L_6^{(0,0)} = span\{X_6^9, X_6^{12}, X_6^{19}, X_6^{22}, X_6^{31}, X_6^{33}, X_6^{45}, X_6^{50}, X_6^{52}, X_6^{60}, X_6^{65}, X_6^{67}, X_6^{90}, X_6^{104}\}$,

$L_6^{(2,-1)} = span\{X_6^6, X_6^{10}, X_6^{11}, X_6^{30}, X_6^{44}, X_6^{57}, X_6^{59}, X_6^{61}, X_6^{87}, X_6^{101}\}$,

$L_6^{(-1,2)} = span\{X_6^3, X_6^8, X_6^{16}, X_6^{32}, X_6^{42}, X_6^{47}, X_6^{49}, X_6^{64}, X_6^{89}, X_6^{103}\}$.

We obtain the natural actions of x_1 and x_2 on $L_6^{(0,0)}$:

$$
\begin{array}{lll}
x_1 \cdot X_6^9 = X_6^6, & x_1 \cdot X_6^{12} = X_6^{11}, & x_1 \cdot X_6^{19} = 2X_6^{10} + X_6^6, \\
x_1 \cdot X_6^{22} = X_6^{11}, & x_1 \cdot X_6^{31} = X_6^{30}, & x_1 \cdot X_6^{33} = X_6^{30}, \\
x_1 \cdot X_6^{45} = X_6^{44}, & x_1 \cdot X_6^{50} = X_6^{44}, & x_1 \cdot X_6^{52} = 0, \\
x_1 \cdot X_6^{60} = X_6^{57} + X_6^{59}, & x_1 \cdot X_6^{65} = 2X_6^{59} + X_6^{30}, & x_1 \cdot X_6^{67} = X_6^{61}, \\
x_1 \cdot X_6^{90} = X_6^{87}, & x_1 \cdot X_6^{104} = 2X_6^{101} + X_6^6 + X_6^{61}, & x_2 \cdot X_6^9 = -X_6^3 + X_6^8, \\
x_2 \cdot X_6^{12} = X_6^3 + X_6^8, & x_2 \cdot X_6^{19} = X_6^{16}, & x_2 \cdot X_6^{22} = X_6^{16}, \\
x_2 \cdot X_6^{31} = 0, & x_2 \cdot X_6^{33} = X_6^{32}, & x_2 \cdot X_6^{45} = X_6^{42}, \\
x_2 \cdot X_6^{50} = X_6^{47} + X_6^{49}, & x_2 \cdot X_6^{52} = 2X_6^{49} - X_6^{32}, & x_2 \cdot X_6^{60} = X_6^{42}, \\
x_2 \cdot X_6^{65} = X_6^{47} + X_6^{64}, & x_2 \cdot X_6^{67} = X_6^{49} + X_6^{64}, & x_2 \cdot X_6^{90} = 2X_6^{89} + X_6^{42}, \\
x_2 \cdot X_6^{104} = X_6^{89} + X_6^{103}. & &
\end{array}
$$

We form a 10×14 matrix A_1 representing $x_1 : L_6^{(0,0)} \longrightarrow L_6^{(2,-1)}$ and a 10×14 matrix A_2 representing $x_2 : L_6^{(0,0)} \longrightarrow L_6^{(-1,2)}$. We combine them vertically to get a 20×14 matrix C. $Rank(C) = 14$ and Rank-Nullity Theorem imply $kernel(C) = \varnothing$.

So the intersection of mapping x_1 and x_2 is empty. We conclude that there is no Lie invariant of degree 6.

In degree 7, there are no Lie invariants in degree 7 since no Hall word of degree 7 has weight $(0,0)$.

In degree 8, there are no Lie invariants in degree 8 since no Hall word of degree 8 has weight $(0,0)$.

From the discussion above, we obtain that $L_i^{(0,0)} \neq \varnothing$ when $i = 3, 6$. Actually, this is true for all $i = 3n$ where n is a positive integer. Moreover, we don't find any Lie invariant in degree 3 and 6. This agrees with Wever's observation.

5.3 Degree 9

In this section, we compute the Lie invariants of degree 9 in the natural representation of $sl(3)$.

Theorem 5.3.1. Any nonzero linear combination of the following 4 homogeneous Lie polynomials is a primitive Lie invariant of degree 9 in the natural representation L of $sl(3)$ where L is the free Lie algebra generated by a, b, c:

$$I_9^1 = X_9^{39} - X_9^{75} + X_9^{111} - X_9^{145} - X_9^{200} + X_9^{236} + X_9^{253} - X_9^{272} - X_9^{343} + X_9^{379},$$

$$\begin{aligned} I_9^2 = \; & X_9^{18} - X_9^{35} + X_9^{39} - X_9^{45} + X_9^{50} - X_9^{69} - X_9^{75} + X_9^{86} - X_9^{102} + 2X_9^{117} - X_9^{122} \\ & + 2X_9^{128} - 2X_9^{134} + X_9^{139} - 3X_9^{145} + 2X_9^{150} - X_9^{155} - X_9^{179} + X_9^{196} - 2X_9^{200} + X_9^{206} \\ & - X_9^{211} - X_9^{225} + 2X_9^{230} - X_9^{236} + X_9^{242} - 2X_9^{247} + 3X_9^{253} - X_9^{258} + 2X_9^{263} - X_9^{278} \\ & + X_9^{295} - X_9^{311} - X_9^{336} + X_9^{353} - X_9^{370} + X_9^{384} - X_9^{401} + X_9^{418}, \end{aligned}$$

$$\begin{aligned} I_9^3 = \; & -X_9^{1040} + X_9^{1047} - X_9^{1051} + 2X_9^{1053} + X_9^{1062} + X_9^{1067} - 2X_9^{1069} - X_9^{1076} + X_9^{1120} - X_9^{1136} \\ & - X_9^{1142} + X_9^{1152} + 2X_9^{1163} - X_9^{1165} + X_9^{1182} - X_9^{1198} - 2X_9^{1203} + X_9^{1205} + X_9^{1212} + 2X_9^{1219} \\ & - X_9^{1221} - X_9^{1233} - X_9^{1263} + X_9^{1279} - X_9^{1283} + 2X_9^{1285} - X_9^{1295} - X_9^{1306} + X_9^{1323} - 2X_9^{1325} \\ & - X_9^{1332} - X_9^{1339} + 2X_9^{1341} + X_9^{1346} + X_9^{1353} - X_9^{1362} + X_9^{1396} - X_9^{1412} - X_9^{1417} + X_9^{1433}, \end{aligned}$$

$$I_9^4 = X_9^{1040} - X_9^{1047} - 2X_9^{1051} + X_9^{1053} - X_9^{1062} + X_9^{1067} + X_9^{1069} - X_9^{1076} - X_9^{1082} + X_9^{1089}.$$

Proof. We regard L_1 as a copy of the simple highest weight $sl(3)$-module with highest weight $(1,-1)$. The natural action of $sl(3)$ is listed in Table 5.2. This action extends to all of L by linearity and the Leibniz rule for derivations.

We want to find elements $Z \in L_9$ which satisfies $D \cdot Z = 0$ for all $D \in sl(3)$. But L_9 is finite dimensional, it decomposes as a direct sum of simple highest weight modules, and it suffices to impose the condition $X \cdot Z = 0$ $(X \in span\{x_1, x_2, x_3\})$ and $H \cdot Z = 0$ $(H \in span\{h_1, h_2\})$. We compute the vector in the intersection of the kernel $x_1 : L_9^{(0,0)} \longrightarrow L_9^{(2,-1)}$ and the kernel $x_2 : L_9^{(0,0)} \longrightarrow L_9^{(-1,2)}$. First, let's list the subscript set of the weight spaces $L_9^{(0,0)}$, $L_9^{(2,-1)}$ and $L_9^{(-1,2)}$:

$$
\begin{aligned}
B_9^{(0,0)} = \{&18, 35, 39, 45, 50, 69, 75, 81, 86, 102, 111, 117, 122, 128, 134, 139, 145, 150, 155,\\
&179, 196, 200, 206, 211, 219, 225, 230, 236, 242, 247, 253, 258, 263, 272, 278, 283,\\
&295, 311, 336, 343, 348, 353, 370, 379, 384, 389, 401, 418, 468, 485, 501, 507, 513,\\
&518, 524, 530, 535, 570, 577, 582, 587, 599, 647, 664, 669, 675, 680, 686, 692, 697,\\
&709, 732, 739, 744, 749, 761, 820, 832, 880, 885, 902, 907, 916, 921, 924, 926, 933,\\
&941, 944, 946, 970, 981, 987, 1000, 1002, 1008, 1040, 1047, 1051, 1053, 1062, 1067,\\
&1069, 1076, 1082, 1089, 1120, 1136, 1142, 1152, 1159, 1163, 1165, 1182, 1198, 1203,\\
&1205, 1212, 1219, 1221, 1226, 1233, 1263, 1272, 1279, 1283, 1285, 1295, 1306, 1318,\\
&1323, 1325, 1332, 1339, 1341, 1346, 1353, 1362, 1396, 1412, 1417, 1433, 1496, 1503,\\
&1518, 1523, 1525, 1530, 1556, 1561, 1608, 1615, 1630, 1635, 1637, 1642, 1668, 1673,\\
&1745, 1754, 1761, 1772, 1781, 1789, 1800, 1808, 1834, 1845, 1849, 1862, 1866, 1871,\\
&1888, 1899, 1903, 1916, 1920, 1925, 1987, 1998, 2000, 2047, 2058, 2060, 2123, 2158\},
\end{aligned}
$$

$$
\begin{aligned}
B_9^{(2,-1)} = \{&15, 21, 27, 32, 48, 84, 120, 127, 132, 137, 154, 165, 171, 176, 182, 188, 193, 199,\\
&204, 209, 228, 235, 240, 245, 262, 271, 276, 281, 293, 310, 352, 388, 400, 450, 465,\\
&471, 477, 482, 516, 523, 528, 533, 586, 598, 629, 633, 639, 644, 650, 656, 661, 678,\\
&685, 690, 695, 707, 748, 760, 878, 895, 900, 903, 905, 914, 925, 931, 937, 939, 945,\\
&986, 1001, 1038, 1043, 1045, 1052, 1068, 1081, 1112, 1118, 1134, 1150, 1155, 1157,\\
&1164, 1204, 1220, 1225, 1248, 1255, 1259, 1261, 1270, 1275, 1277, 1284, 1291, 1293,\\
&1298, 1305, 1324, 1340, 1345, 1361, 1480, 1494, 1499, 1501, 1524, 1529, 1592, 1599,\\
&1606, 1611, 1613, 1618, 1636, 1641, 1744, 1753, 1760, 1771, 1780, 1788, 1833, 1844,
\end{aligned}
$$

$1848, 1882, 1887, 1891, 1898, 1902, 1907, 1978, 1986, 1988, 2038, 2046, 2048, 2118, 2153\},$

$$
\begin{aligned}
B_9^{(-1,2)} = \{&17, 34, 38, 44, 49, 57, 63, 68, 74, 80, 85, 91, 96, 101, 110, 116, 121, 133, 149, \\
&178, 205, 218, 224, 229, 241, 257, 277, 325, 330, 335, 347, 364, 383, 467, 484, 489, \\
&495, 500, 506, 512, 517, 529, 552, 559, 564, 569, 581, 646, 668, 674, 679, 691, 721, \\
&726, 731, 743, 802, 814, 872, 879, 882, 884, 901, 915, 918, 920, 943, 967, 969, 975, \\
&999, 1039, 1050, 1059, 1061, 1066, 1073, 1119, 1128, 1135, 1139, 1141, 1151, 1162, \\
&1174, 1179, 1181, 1188, 1195, 1197, 1202, 1209, 1218, 1271, 1282, 1315, 1317, 1322, \\
&1329, 1338, 1388, 1393, 1409, 1488, 1495, 1510, 1515, 1517, 1522, 1548, 1553, 1607, \\
&1627, 1629, 1634, 1660, 1665, 1742, 1751, 1759, 1779, 1798, 1807, 1828, 1839, 1843, \\
&1856, 1860, 1865, 1897, 1914, 1919, 1984, 1995, 1997, 2044, 2055, 2057, 2122, 2157\}.
\end{aligned}
$$

We compute the natural action of x_1 on each Hall word X_9^i with $i \in B_9^{(0,0)}$. All 186 equations are listed in Appendix E.4. We use those equations to form a 140×186 matrix C_1 to represent the map $x_1 : L_9^{(0,0)} \longrightarrow L_9^{(2,-1)}$.

We next compute the natural action of x_2 on each Hall word X_9^i with $i \in B_9^{(0,0)}$. All 186 equations are listed in Appendix E.4. We use those equations to form a 140×186 matrix C_2 to represent the map $x_2 : L_9^{(0,0)} \longrightarrow L_9^{(-1,2)}$.

At last, we merge C_1 and C_2 vertically to get a 280×186 matrix C and compute $Nullity(C) = 4$ since $Rank(C) = 182$. NullSpace in MAPLE can compute the basis of the kernel which are

$$I_9^1 = X_9^{39} - X_9^{75} + X_9^{111} - X_9^{145} - X_9^{200} + X_9^{236} + X_9^{253} - X_9^{272} - X_9^{343} + X_9^{379},$$

$$
\begin{aligned}
I_9^2 = {} & X_9^{18} - X_9^{35} + X_9^{39} - X_9^{45} + X_9^{50} - X_9^{69} - X_9^{75} + X_9^{86} - X_9^{102} + 2X_9^{117} - X_9^{122} \\
& + 2X_9^{128} - 2X_9^{134} + X_9^{139} - 3X_9^{145} + 2X_9^{150} - X_9^{155} - X_9^{179} + X_9^{196} - 2X_9^{200} + X_9^{206} \\
& - X_9^{211} - X_9^{225} + 2X_9^{230} - X_9^{236} + X_9^{242} - 2X_9^{247} + 3X_9^{253} - X_9^{258} + 2X_9^{263} - X_9^{278} \\
& + X_9^{295} - X_9^{311} - X_9^{336} + X_9^{353} - X_9^{370} + X_9^{384} - X_9^{401} + X_9^{418},
\end{aligned}
$$

$$
\begin{aligned}
I_9^3 = {} & -X_9^{1040} + X_9^{1047} - X_9^{1051} + 2X_9^{1053} + X_9^{1062} + X_9^{1067} - 2X_9^{1069} - X_9^{1076} + X_9^{1120} - X_9^{1136} \\
& - X_9^{1142} + X_9^{1152} + 2X_9^{1163} - X_9^{1165} + X_9^{1182} - X_9^{1198} - 2X_9^{1203} + X_9^{1205} + X_9^{1212} + 2X_9^{1219}
\end{aligned}
$$

$$-X_9^{1221}-X_9^{1233}-X_9^{1263}+X_9^{1279}-X_9^{1283}+2X_9^{1285}-X_9^{1295}-X_9^{1306}+X_9^{1323}-2X_9^{1325}$$
$$-X_9^{1332}-X_9^{1339}+2X_9^{1341}+X_9^{1346}+X_9^{1353}-X_9^{1362}+X_9^{1396}-X_9^{1412}-X_9^{1417}+X_9^{1433},$$

$$I_9^4 = X_9^{1040}-X_9^{1047}-2X_9^{1051}+X_9^{1053}-X_9^{1062}+X_9^{1067}+X_9^{1069}-X_9^{1076}-X_9^{1082}+X_9^{1089}.$$

This completes the proof. □

Remark 5.3.1. We can use Theorem 2.8.2 to compute the Lie invariants too. First, we merge C_1 and C_2 vertically to obtain a 280×186 matrix C. Then we compute the Hermite normal form H and the unimodular transform matrix U of the transpose of C. Since $Rank(H) = 182$ and U is a 186×186 nonsingular matrix, the last $186 - 182 = 4$ rows of U form the null space of C. In fact, we obtain exactly the same 4 Lie invariants by MAPLE as we see before. Thus, they are the basis vectors for K_9 and J_9.

Remark 5.3.2. Since the lowest degree Lie invariants in the natural representation of $sl(3)$ are of degree 9 and there are more than 2 of them, the lowest degree non-primitive Lie invariants are of degree 18.

5.4 Degree 12

There are 44220 Hall words of degree 12 in 3 generators. We ignore many details since monotonous repetitive works are unnecessary. We obtain the main result in degree 12 as follows:

Theorem 5.4.1. Any nonzero linear combination of the 35 homogeneous Lie polynomials in Appendix F is a primitive Lie invariant of degree 12 in the natural representation L of $sl(3)$ where L is the free Lie algebra generated by a, b, c.

Proof. We apply MAPLE command to compute $B_{12}^{(0,0)}$, $B_{12}^{(2,-1)}$ and $B_{12}^{(-1,2)}$. Then we use MAPLE command to determine the 2130 action identities for the map $L_{12}^{(0,0)} \longrightarrow L_{12}^{(2,-1)}$ and other 2130 action identities for the map $L_{12}^{(0,0)} \longrightarrow L_{12}^{(-1,2)}$. Then we form two 2310×2880 matrices and merge them together vertically as we did in degree 9 case. A basis for the nullspace of the 4620×2880 matrix gives the Lie invariants of degree 12. We ignore lots of details here due to potential large space usage. □

Chapter 6

Conclusion

We have discussed these three cases in previous chapters:

1. Lie invariants of degree up to 14 in the natural representation of $sl(2)$.

2. Lie invariants of degree up to 9 in the adjoint representation of $sl(2)$.

3. Lie invariants of degree up to 12 in the natural representation of $sl(3)$.

In fact, this thesis involves many Computer Algebra problems. We can't compute the invariant Lie polynomials before reaching some high and finite degree due to limitations of computer hardware by the standard methods we used in previous chapters. In higher degree case, since the entries in the transformation matrix which represents the map between weight spaces may be very large, we may apply the Lattice Basis Reduction algorithm (see [8], §2.6) to reduce the large entries to small ones. This may greatly improve the speed of calculation for the nullspace. But we do not include it in this thesis since the entries we have till now are at most two digital decimal numbers which is small in a sense. We can also improve the Hermite Normal Form algorithm. The Algorithm 2.4.1 is not of polynomial time since it is basically a modification of Gaussian elimination algorithm. But we still use it in this thesis since the matrices we have contain very small integer entries and the Algorithm 2.4.1 works well in those cases. Nowadays, a very active area in Computer Algebra is to search the time and space efficient HNF algorithm. A classical polynomial HNF algorithm is introduced by Kannan and Bachem [16]. A popular linear space cost HNF algorithm for random matrix is Micciancio and Warinschi [21].

The classification of simple Lie algebras and their representations is well-knows. There are infinitely many different simple finite dimensional Lie algebras, and each of them has infinitely many distinct irreducible representations. So the computational problem is to determine the invariant Lie polynomials generated by these representations. i.e. we can compute Lie invariants by considering the bigger representation of $sl(2)$ and $sl(3)$, dimensions 4 and 5, etc, and simple finite Lie algebras of higher rank such as $sl(n)$ with $n > 3$. To find a general characterization of the invariants is a theoretical problem rather than a computational problem. It is known that, in classical problems of invariants in (associative or/and commutative) polynomials, all of the invariants are generated by finite sets. But in this thesis, the Lie (nonassociative) invariants we considered seem as though they are not finitely generated. It looks as though, as the degree approaches infinity, the number of Lie invariants approaches infinity as well. But until now, there is no such result proving it. So we conclude with the following conjecture:

Conjecture 6.0.1. There exist infinitely many primitive invariant Lie polynomials in the three cases listed above.

Furthermore, we can consider the question of whether it is possible to prove asymptotic estimates for the number of primitive invariants in the theory of free Lie algebras. Thus, it is an open-ended problem.

This thesis is an attempt to extend classical (associative, commutative) invariant theory to the nonassociative case, in particular to Lie invariants in free Lie algebras. This is the pure mathematical motivation for the research.

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Appendix A

Maple programs

A.1 Hall Words

The following MAPLE program is to generate Hall words in 2 or 3 generators. It strictly follows Marshell Hall's definitions of Hall words. See [10] for details.

```
with(ListTools):

with(LinearAlgebra):

maxdeg := 10; # the degree of Hall words

Halldeg[1] := [ a, b ];  # 2 generators case

# Halldeg[1] := [ a, b, c ]; # 3 generators case

Hallset := Halldeg[1]:

for deg from 2 to maxdeg do
  Halldeg[deg] := []:
  for q from ceil(deg/2) to deg-1 do
    p := deg-q:
    if p = q then
      for i from 2 to nops(Halldeg[p]) do
        for j from 1 to i-1 do
          Halldeg[deg] := [ op(Halldeg[deg]),
              [Halldeg[p][i],Halldeg[p][j]] ]:
        od:
      od:
    else
      for x in Halldeg[q] do
        for y in Halldeg[p] do
          if Search(x[2],Hallset) <= Search(y,Hallset) then
            Halldeg[deg] := [ op(Halldeg[deg]), [x,y] ]:
          fi:
        od:
      od:
    fi:
  od:
  Hallset := [ op(Hallset), op(Halldeg[deg]) ]:
```

```
od:
```

A.2 Standard Form

The following program is the MAPLE implementation of algorithm 2.4.1 which transforms any elements in free magma into an integer linear combination of Hall words. This MAPLE implementation is done by Murray Bremner.

```
compare := proc( x, y )
  local result:
  if x[1] > y[1] then result := true fi:
  if x[1] < y[1] then result := false fi:
  if x[1] = y[1] then result := evalb( x[2] > y[2] ) fi:
  RETURN( result )
end:

position := proc( x )
  local i:
  global Hallset:
  i := 1:
  while Hallset[i] <> x do i := i + 1 od:
  RETURN( i )
end:

compress := proc( inputlist )
  local coef, inputcopy, inputcopycopy, outputlist, term, termtwo:
  outputlist := []:
  inputcopy := copy(inputlist):
  while inputcopy <> [] do
    term := inputcopy[1]:
    coef := 0:
    for termtwo in inputcopy do
      if term[2] = termtwo[2] then
        coef := coef + termtwo[1]
      fi
    od:
    if coef <> 0 then
      outputlist := [ op(outputlist), [ coef, term[2] ] ]
    fi:
    inputcopycopy := []:
    for termtwo in inputcopy do
      if termtwo[2] <> term[2] then
        inputcopycopy := [ op(inputcopycopy), termtwo ]
      fi
    od:
```

```
    inputcopy := inputcopycopy
  od:
  RETURN( outputlist )
end:

standardform := proc( x ) option remember: # Algorithm 2.4.1
  local a, b, au, bv, newresult, result, signchange,
        temp, term, termtwo, u, v, w, y, ys, z, zs:
  if nops(x) = 1 then
    result := [ [ 1, x ] ]
  else
    result := []:
    # Hall's first step
    y := x[1]:
    z := x[2]:
    ys := standardform( y ):
    zs := standardform( z ):
    for au in ys do
      a,u := op(au):
      for bv in zs do
        b,v := op(bv):
        # Hall's second step
        signchange := 1:
        if position(u) < position(v) then
          signchange := -signchange:
          temp := u:
          u := v:
          v := temp
        fi:
        # Hall's third step
        if position(u) > position(v) then
          if nops(u) = 1 then
            result := [ op(result), [ signchange*a*b, [u,v] ] ]
          else
            z,w := op(u):
            if position(v) >= position(w) then
              result := [ op(result),
                [ signchange*a*b, [[z,w],v] ]
                ]
            else
              result := [ op(result),
                [ -signchange*a*b, [[w,v],z] ],
                [ signchange*a*b, [[z,v],w] ] ]
             fi
           fi
```

```
        fi
      od
    od
  fi:
   result := compress( result ):
   # Hall's fourth step
   newresult := []:
   for term in result do
     if member( term[2], Hallset ) then
       newresult := [ op(newresult), term ]
      else
       for termtwo in standardform( term[2] ) do
       newresult := [op(newresult),[term[1]*termtwo[1],termtwo[2]]]
      od
     fi
  od:
  RETURN( newresult )
end:
```

A.3 Witt Formula

We use following MAPLE program to implement the Witt formula in theorem 2.5.1 which is to compute $\dim L_n$:

```
# n denotes the degree and d denotes the number of generators.
dimL := proc( n , d )
  local f, result:
  result := 0:
    for f in numtheory[divisors](n) do
      result := result + numtheory[mobius](f)*d^(n/f):
    od:
  result := result / n:
  RETURN( result ):
end:
```

The following MAPLE program is to implement the Generalized Witt formula in theorem 2.5.3 which is to compute $\dim K_n$ in 2 generators case:

```
dimK := proc( n )
  local f, result:
  result := 0:
  if n mod 2 = 0 then
    for f in numtheory[divisors](n) do
      result := result + numtheory[mobius](f)*
              combinat[Chi]([n/2,n/2],[seq(f,i=1..n/f)])
```

```
    od:
    result := result / n
  fi:
  RETURN( result )
end:
```

For 3 generators case, we have

```
dimK := proc( n )
  local f, result:
  result := 0:
  if n mod 3 = 0 then
    for f in numtheory[divisors](n) do
      result := result + numtheory[mobius](f)*
                combinat[Chi]([n/3,n/3,n/3],[seq(f,i=1..n/f)])
    od:
    result := result / n
  fi:
  RETURN( result )
end:
```

To compute $\dim J_n$, we use following MAPLE program (up to degree 20 and 2 generators case)

```
restart:

with(combinat):

d := table():

for i to 20 do d[i]:=0 od:

w := proc( p )
  local dproduct, i, innersum, j, result:
  global d:
  result := 0:
  for j to p do
    innersum := 0:
    for i in composition(p,j) do
      dproduct := mul( d[i[j]], j=1..nops(i) ):
      innersum := innersum + dproduct
    od:
    result := result + (1/j)*innersum
  od:
  RETURN( p * result )
end:
```

```
dimKprime := proc( n )
  local f, result:
  result := 0:
    for f in numtheory[divisors](n) do
      result := result + numtheory[mobius](f)*w( n/f ):
    od:
  result := result / n:
  RETURN( result ):
end:

dimK := proc( n )
  local f, result:
  result := 0:
  if n mod 2 = 0 then
    for f in numtheory[divisors](n) do
      result := result + numtheory[mobius](f)*
                combinat[Chi]([n/2,n/2],[seq(f,i=1..n/f)])
    od:
    result := result / n
  fi:
  RETURN( result )
end:

for i from 1 to 20  do
  if dimK(i) <> dimKprime(i) then
     d[i]:= dimK(i)-dimKprime(i):
  fi:
od:
```

A.4 Hermite Normal Form

In this section, we present a MAPLE implementation of Hermite normal form computing algorithm 2.7.1.

```
# Modified Gaussian elimination to compute Hermite normal form.
with(LinearAlgebra):

identitymatrix := proc( k )
 local A, i:
 i:=copy(k):
 A:=Matrix(i,i,(i,j)-> if i=j then 1 else 0 fi ):
 RETURN( A ):
end:
```

```
quotient:=proc( x , y ) # return a quotient of integer x and y
                          such that the remainder is positive.
  local x1, x2, q , r:
  x1:=copy(x): x2:=copy(y):
  r:=( x1 mod x2 ):
  q:=(x1-r)/x2 :
 RETURN( q ):
end:

absmin:=proc( A , a , b, c)
  local col, H, i, l, u, vectorlist:
  l:=copy(a):
  u:=copy(b):
  col:=copy(c):
  H:=copy(A):
  vectorlist:=sort(convert(abs(H[a..b,col]),list)):
  i:=1:
  while vectorlist[i]=0 and i <= nops(vectorlist) do
     i:=i+1:
  od:
 RETURN( vectorlist[i] ):
end:

zerocolumn:=proc( M, I1, I2, J )
  local H, i, i1, i2, j:
  H := copy( M ):
  i1 := copy( I1 ):
  i2 := copy( I2 ):
  j := copy( J ):
  i:=i1:
  while i <= i2 and H[i,j] = 0 do
    i:=i+1:
  od:
  if i = i2+1 then
    RETURN( true ):
  else
    RETURN( false ):
  fi:
end:

HermiteNormalForm := proc( A )
  local H, i, j, k, m, n, p, q, s, U:
   H:=copy(A):
   m:=RowDimension( H ):
   n:=ColumnDimension( H ):
```

```
  U:=identitymatrix( m ):
   i:=1:  j:=1:
   while i <= m and j <= n do
      if zerocolumn(H,i,m,j)=true then
       j:=j+1:
      else
       while H[i,j] <= 0 or zerocolumn(H,i+1,m,j)=false  do
          s:=absmin( H, i, m, j ):
          p:=copy(i):
          while abs(H[p,j]) <> s and p <= m do
            p:=p+1:
          od:
          H:=RowOperation(H, [i,p]):
          U:=RowOperation(U, [i,p]):
          if H[i,j]<0 then
            H:=RowOperation(H, i, -1):
            U:=RowOperation(U, i, -1):
          fi:
          for k from i+1 to m do
             q:=quotient( H[k,j], H[i,j] ):
             if q <> 0 then
               H:=RowOperation( H, [k, i], -q ):
               U:=RowOperation( U, [k, i], -q ):
             fi:
          od:
        od:
        for k from 1 to i-1 do
            q:=quotient( H[k,j], H[i,j] ):
             if q <> 0 then
               H:=RowOperation( H, [k, i], -q ):
               U:=RowOperation( U, [k, i], -q ):
             fi:
        od:
      i:=i+1:
      j:=j+1:
     fi:
   od:
  RETURN( [H , U] ):
end:
```

Appendix B

Table of Dimensions

n	$\dim L_n$	$\dim K_n$	$\dim J_n$
1	3	.	.
2	3	.	.
3	8	.	.
4	18	.	.
5	48	.	.
6	116	.	.
7	312	.	.
8	810	.	.
9	2,184	4	4
10	5,880	.	.
11	16,106	.	.
12	44,220	35	35
13	122,640	.	.
14	341,484	.	.
15	956,576	398	398
16	2,690,010	.	.
17	7,596,480	.	.
18	21,522,228	4834	4828
19	61,171,656	.	.
20	174,336,264	.	.

Table B.1: Dimensions of Space of Invariants in 3 generators

n	$\dim L_n$	$\dim K_n$	$\dim J_n$
1	2	.	.
2	1	1	1
3	2	.	.
4	3	.	.
5	6	.	.
6	9	1	1
7	18	.	.
8	30	1	.
9	56	.	.
10	99	5	4
11	186	.	.
12	335	9	4
13	630	.	.
14	1,161	33	23
15	2,182	.	.
16	4.080	85	48
17	7,710	.	.
18	14,532	276	182
19	27,594	.	.
20	52,377	827	513
21	99,858	.	.
22	190,557	2,693	1,755
23	364,722	.	.
24	698,870	8,626	5,539
25	1,342,176	.	.
26	2,580,795	28,639	18,764
27	4,971,008	.	.
28	9,586,395	95,393	62,455
29	18,512,790	.	.
30	35,790,267	323,367	213,677

Table B.2: Dimensions of Space of Invariants in 2 generators

Appendix C

Hall words of 2 generators

C.1 Degree 5

$$[[[b,a],a],[b,a]]_1, \quad [[[b,a],b],[b,a]]_2, \quad [[[[b,a],a],a],a]_3,$$
$$[[[[b,a],a],a],b]_4, \quad [[[[b,a],a],b],b]_5, \quad [[[[b,a],b],b],b]_6.$$

C.2 Degree 6

$$[[[b,a],b],[[b,a],a]]_1, \ [[[[b,a],a],a],[b,a]]_2, \ [[[[b,a],a],b],[b,a]]_3,$$
$$[[[[b,a],b],b],[b,a]]_4, \ [[[[[b,a],a],a],a],a]_5, \ [[[[[b,a],a],a],a],b]_6,$$
$$[[[[[b,a],a],a],b],b]_7, \ [[[[[b,a],a],b],b],b]_8, \ [[[[[b,a],b],b],b],b]_9.$$

C.3 Degree 7

$$[[[[b,a],a],a],[[b,a],a]]_1, \quad [[[[b,a],a],a],[[b,a],b]]_2, \ [[[[b,a],a],b],[[b,a],a]]_3,$$
$$[[[[b,a],a],b],[[b,a],b]]_4, \quad [[[[b,a],b],b],[[b,a],a]]_5, \ [[[[b,a],b],b],[[b,a],b]]_6,$$
$$[[[[b,a],a],[b,a]],[b,a]]_7, \quad [[[[b,a],b],[b,a]],[b,a]]_8, \ [[[[[b,a],a],a],a],[b,a]]_9,$$
$$[[[[[b,a],a],a],b],[b,a]]_{10}, \quad [[[[[b,a],a],b],b],[b,a]]_{11}, \ [[[[[b,a],b],b],b],[b,a]]_{12},$$
$$[[[[[[b,a],a],a],a],a],a]_{13}, \ [[[[[[b,a],a],a],a],a],b]_{14}, \ [[[[[[b,a],a],a],a],b],b]_{15},$$
$$[[[[[[b,a],a],a],b],b],b]_{16}, \quad [[[[[[b,a],a],b],b],b],b]_{17}, \ [[[[[[b,a],b],b],b],b],b]_{18}.$$

C.4 Degree 8

$$[[[[b,a],a],b],[[[b,a],a],a]]_1, \quad [[[[b,a],b],b],[[[b,a],a],a]]_2,$$
$$[[[[b,a],b],b],[[[b,a],a],b]]_3, \quad [[[[b,a],a],[b,a]],[[b,a],a]]_4,$$
$$[[[[b,a],a],[b,a]],[[b,a],b]]_5, \quad [[[[b,a],b],[b,a]],[[b,a],a]]_6,$$
$$[[[[b,a],b],[b,a]],[[b,a],b]]_7, \quad [[[[[b,a],a],a],a],[[b,a],a]]_8,$$
$$[[[[[b,a],a],a],a],[[b,a],b]]_9, \quad [[[[[b,a],a],a],b],[[b,a],a]]_{10},$$
$$[[[[[b,a],a],a],b],[[b,a],b]]_{11}, \quad [[[[[b,a],a],b],b],[[b,a],a]]_{12},$$
$$[[[[[b,a],a],b],b],[[b,a],b]]_{13}, \quad [[[[[b,a],b],b],b],[[b,a],a]]_{14},$$
$$[[[[[b,a],b],b],b],[[b,a],b]]_{15}, \quad [[[[[b,a],a],a],[b,a]],[b,a]]_{16},$$

$[[[[[b,a],a],b],[b,a]],[b,a]]_{17}$, $[[[[[b,a],b],b],[b,a]],[b,a]]_{18}$,
$[[[[[[b,a],a],a],a],a],[b,a]]_{19}$, $[[[[[[b,a],a],a],a],b],[b,a]]_{20}$,
$[[[[[[b,a],a],a],b],b],[b,a]]_{21}$, $[[[[[[b,a],a],b],b],b],[b,a]]_{22}$,
$[[[[[[b,a],b],b],b],b],[b,a]]_{23}$, $[[[[[[[b,a],a],a],a],a],a],a]_{24}$,
$[[[[[[[b,a],a],a],a],a],a],b]_{25}$, $[[[[[[[b,a],a],a],a],a],b],b]_{26}$,
$[[[[[[[b,a],a],a],a],b],b],b]_{27}$, $[[[[[[[b,a],a],a],b],b],b],b]_{28}$,
$[[[[[[[b,a],a],b],b],b],b],b]_{29}$, $[[[[[[[b,a],b],b],b],b],b],b]_{30}$.

C.5 Degree 9

$[[[[b,a],a],[b,a]],[[[b,a],a],a]]_1$, $[[[[b,a],a],[b,a]],[[[b,a],a],b]]_2$,
$[[[[b,a],a],[b,a]],[[[b,a],b],b]]_3$, $[[[[b,a],b],[b,a]],[[[b,a],a],a]]_4$,
$[[[[b,a],b],[b,a]],[[[b,a],a],b]]_5$, $[[[[b,a],b],[b,a]],[[[b,a],b],b]]_6$,
$[[[[[b,a],a],a],a],[[[b,a],a],a]]_7$, $[[[[[b,a],a],a],a],[[[b,a],a],b]]_8$,
$[[[[[b,a],a],a],a],[[[b,a],b],b]]_9$, $[[[[[b,a],a],a],b],[[[b,a],a],a]]_{10}$,
$[[[[[b,a],a],a],b],[[[b,a],a],b]]_{11}$, $[[[[[b,a],a],a],b],[[[b,a],b],b]]_{12}$,
$[[[[[b,a],a],b],b],[[[b,a],a],a]]_{13}$, $[[[[[b,a],a],b],b],[[[b,a],a],b]]_{14}$,
$[[[[[b,a],a],b],b],[[[b,a],b],b]]_{15}$, $[[[[[b,a],b],b],b],[[[b,a],a],a]]_{16}$,
$[[[[[b,a],b],b],b],[[[b,a],a],b]]_{17}$, $[[[[[b,a],b],b],b],[[[b,a],b],b]]_{18}$,
$[[[[b,a],b],[[b,a],a]],[[b,a],a]]_{19}$, $[[[[b,a],b],[[b,a],a]],[[b,a],b]]_{20}$,
$[[[[[b,a],a],a],[b,a]],[[b,a],a]]_{21}$, $[[[[[b,a],a],a],[b,a]],[[b,a],b]]_{22}$,
$[[[[[b,a],a],b],[b,a]],[[b,a],a]]_{23}$, $[[[[[b,a],a],b],[b,a]],[[b,a],b]]_{24}$,
$[[[[[b,a],b],b],[b,a]],[[b,a],a]]_{25}$, $[[[[[b,a],b],b],[b,a]],[[b,a],b]]_{26}$,
$[[[[[[b,a],a],a],a],a],[[b,a],a]]_{27}$, $[[[[[[b,a],a],a],a],a],[[b,a],b]]_{28}$,
$[[[[[[b,a],a],a],a],b],[[b,a],a]]_{29}$, $[[[[[[b,a],a],a],a],b],[[b,a],b]]_{30}$,
$[[[[[[b,a],a],a],b],b],[[b,a],a]]_{31}$, $[[[[[[b,a],a],a],b],b],[[b,a],b]]_{32}$,
$[[[[[[b,a],a],b],b],b],[[b,a],a]]_{33}$, $[[[[[[b,a],a],b],b],b],[[b,a],b]]_{34}$,
$[[[[[[b,a],b],b],b],b],[[b,a],a]]_{35}$, $[[[[[[b,a],b],b],b],b],[[b,a],b]]_{36}$,
$[[[[[b,a],a],[b,a]],[b,a]],[b,a]]_{37}$, $[[[[[b,a],b],[b,a]],[b,a]],[b,a]]_{38}$,
$[[[[[[b,a],a],a],a],[b,a]],[b,a]]_{39}$, $[[[[[[b,a],a],a],b],[b,a]],[b,a]]_{40}$,
$[[[[[[b,a],a],b],b],[b,a]],[b,a]]_{41}$, $[[[[[[b,a],b],b],b],[b,a]],[b,a]]_{42}$,
$[[[[[[[b,a],a],a],a],a],a],[b,a]]_{43}$, $[[[[[[[b,a],a],a],a],a],b],[b,a]]_{44}$,
$[[[[[[[b,a],a],a],a],b],b],[b,a]]_{45}$, $[[[[[[[b,a],a],a],b],b],b],[b,a]]_{46}$,
$[[[[[[[b,a],a],b],b],b],b],[b,a]]_{47}$, $[[[[[[[b,a],b],b],b],b],b],[b,a]]_{48}$,
$[[[[[[[[b,a],a],a],a],a],a],a],a]_{49}$, $[[[[[[[[b,a],a],a],a],a],a],a],b]_{50}$,
$[[[[[[[[b,a],a],a],a],a],a],b],b]_{51}$, $[[[[[[[[b,a],a],a],a],a],b],b],b]_{52}$,
$[[[[[[[[b,a],a],a],a],b],b],b],b]_{53}$, $[[[[[[[[b,a],a],a],b],b],b],b],b]_{54}$,

$$[[[[[[[[b,a],a],b],b],b],b],b],b]_{55}, \qquad [[[[[[[[b,a],b],b],b],b],b],b],b]_{56}.$$

C.6 Degree 10

$$[[[[b,a],b],[b,a]],[[[b,a],a],[b,a]]]_{1}, \qquad [[[[[b,a],a],a],a],[[[b,a],a],[b,a]]]_{2},$$
$$[[[[[b,a],a],a],a],[[[b,a],b],[b,a]]]_{3}, \qquad [[[[[b,a],a],a],b],[[[b,a],a],[b,a]]]_{4},$$
$$[[[[[b,a],a],a],b],[[[b,a],b],[b,a]]]_{5}, \qquad [[[[[b,a],a],a],b],[[[[b,a],a],a],a]]_{6},$$
$$[[[[[b,a],a],b],b],[[[b,a],a],[b,a]]]_{7}, \qquad [[[[[b,a],a],b],b],[[[b,a],b],[b,a]]]_{8},$$
$$[[[[[b,a],a],b],b],[[[[b,a],a],a],a]]_{9}, \qquad [[[[[b,a],a],b],b],[[[[b,a],a],a],b]]_{10},$$
$$[[[[[b,a],b],b],b],[[[b,a],a],[b,a]]]_{11}, \qquad [[[[[b,a],b],b],b],[[[b,a],b],[b,a]]]_{12},$$
$$[[[[[b,a],b],b],b],[[[[b,a],a],a],a]]_{13}, \qquad [[[[[b,a],b],b],b],[[[[b,a],a],a],b]]_{14},$$
$$[[[[[b,a],b],b],b],[[[[b,a],a],b],b]]_{15}, \qquad [[[[b,a],b],[[b,a],a]],[[[b,a],a],a]]_{16},$$
$$[[[[b,a],b],[[b,a],a]],[[[b,a],a],b]]_{17}, \qquad [[[[b,a],b],[[b,a],a]],[[[b,a],b],b]]_{18},$$
$$[[[[[b,a],a],a],[b,a]],[[[b,a],a],a]]_{19}, \qquad [[[[[b,a],a],a],[b,a]],[[[b,a],a],b]]_{20},$$
$$[[[[[b,a],a],a],[b,a]],[[[b,a],b],b]]_{21}, \qquad [[[[[b,a],a],b],[b,a]],[[[b,a],a],a]]_{22},$$
$$[[[[[b,a],a],b],[b,a]],[[[b,a],a],b]]_{23}, \qquad [[[[[b,a],a],b],[b,a]],[[[b,a],b],b]]_{24},$$
$$[[[[[b,a],b],b],[b,a]],[[[b,a],a],a]]_{25}, \qquad [[[[[b,a],b],b],[b,a]],[[[b,a],a],b]]_{26},$$
$$[[[[[b,a],b],b],[b,a]],[[[b,a],b],b]]_{27}, \qquad [[[[[[b,a],a],a],a],a],[[[b,a],a],a]]_{28},$$
$$[[[[[[b,a],a],a],a],a],[[[b,a],a],b]]_{29}, \qquad [[[[[[b,a],a],a],a],a],[[[b,a],b],b]]_{30},$$
$$[[[[[[b,a],a],a],a],b],[[[b,a],a],a]]_{31}, \qquad [[[[[[b,a],a],a],a],b],[[[b,a],a],b]]_{32},$$
$$[[[[[[b,a],a],a],a],b],[[[b,a],b],b]]_{33}, \qquad [[[[[[b,a],a],a],b],b],[[[b,a],a],a]]_{34},$$
$$[[[[[[b,a],a],a],b],b],[[[b,a],a],b]]_{35}, \qquad [[[[[[b,a],a],a],b],b],[[[b,a],b],b]]_{36},$$
$$[[[[[[b,a],a],b],b],b],[[[b,a],a],a]]_{37}, \qquad [[[[[[b,a],a],b],b],b],[[[b,a],a],b]]_{38},$$
$$[[[[[[b,a],a],b],b],b],[[[b,a],b],b]]_{39}, \qquad [[[[[[b,a],b],b],b],b],[[[b,a],a],a]]_{40},$$
$$[[[[[[b,a],b],b],b],b],[[[b,a],a],b]]_{41}, \qquad [[[[[[b,a],b],b],b],b],[[[b,a],b],b]]_{42},$$
$$[[[[[b,a],a],a],[[b,a],a]],[[b,a],a]]_{43}, \qquad [[[[[b,a],a],a],[[b,a],a]],[[b,a],b]]_{44},$$
$$[[[[[b,a],a],a],[[b,a],b]],[[b,a],b]]_{45}, \qquad [[[[[b,a],a],b],[[b,a],a]],[[b,a],a]]_{46},$$
$$[[[[[b,a],a],b],[[b,a],a]],[[b,a],b]]_{47}, \qquad [[[[[b,a],a],b],[[b,a],b]],[[b,a],b]]_{48},$$
$$[[[[[b,a],b],b],[[b,a],a]],[[b,a],a]]_{49}, \qquad [[[[[b,a],b],b],[[b,a],a]],[[b,a],b]]_{50},$$
$$[[[[[b,a],b],b],[[b,a],b]],[[b,a],b]]_{51}, \qquad [[[[[b,a],a],[b,a]],[b,a]],[[b,a],a]]_{52},$$
$$[[[[[b,a],a],[b,a]],[b,a]],[[b,a],b]]_{53}, \qquad [[[[[b,a],b],[b,a]],[b,a]],[[b,a],a]]_{54},$$
$$[[[[[b,a],b],[b,a]],[b,a]],[[b,a],b]]_{55}, \qquad [[[[[[b,a],a],a],a],[b,a]],[[b,a],a]]_{56},$$
$$[[[[[[b,a],a],a],a],[b,a]],[[b,a],b]]_{57}, \qquad [[[[[[b,a],a],a],b],[b,a]],[[b,a],a]]_{58},$$
$$[[[[[[b,a],a],a],b],[b,a]],[[b,a],b]]_{59}, \qquad [[[[[[b,a],a],b],b],[b,a]],[[b,a],a]]_{60},$$
$$[[[[[[b,a],a],b],b],[b,a]],[[b,a],b]]_{61}, \qquad [[[[[[b,a],b],b],b],[b,a]],[[b,a],a]]_{62},$$
$$[[[[[[b,a],b],b],b],[b,a]],[[b,a],b]]_{63}, \qquad [[[[[[[b,a],a],a],a],a],a],[[b,a],a]]_{64},$$
$$[[[[[[[b,a],a],a],a],a],a],[[b,a],b]]_{65}, \qquad [[[[[[[b,a],a],a],a],a],b],[[b,a],a]]_{66},$$

$[[[[[[[b,a],a],a],a],a],b],[[b,a],b]]_{67}$, $[[[[[[[b,a],a],a],a],b],b],[[b,a],a]]_{68}$,
$[[[[[[[b,a],a],a],a],b],b],[[b,a],b]]_{69}$, $[[[[[[[b,a],a],a],b],b],b],[[b,a],a]]_{70}$,
$[[[[[[[b,a],a],a],b],b],b],[[b,a],b]]_{71}$, $[[[[[[[b,a],a],b],b],b],b],[[b,a],a]]_{72}$,
$[[[[[[[b,a],a],b],b],b],b],[[b,a],b]]_{73}$, $[[[[[[[b,a],b],b],b],b],b],[[b,a],a]]_{74}$,
$[[[[[[[b,a],b],b],b],b],b],[[b,a],b]]_{75}$, $[[[[[[b,a],a],a],[b,a]],[b,a]],[b,a]]_{76}$,
$[[[[[[b,a],a],b],[b,a]],[b,a]],[b,a]]_{77}$, $[[[[[[b,a],b],b],[b,a]],[b,a]],[b,a]]_{78}$,
$[[[[[[[b,a],a],a],a],a],[b,a]],[b,a]]_{79}$, $[[[[[[[b,a],a],a],a],b],[b,a]],[b,a]]_{80}$,
$[[[[[[[b,a],a],a],b],b],[b,a]],[b,a]]_{81}$, $[[[[[[[b,a],a],b],b],b],[b,a]],[b,a]]_{82}$,
$[[[[[[[b,a],b],b],b],b],[b,a]],[b,a]]_{83}$, $[[[[[[[[b,a],a],a],a],a],a],a],[b,a]]_{84}$,
$[[[[[[[[b,a],a],a],a],a],a],b],[b,a]]_{85}$, $[[[[[[[[b,a],a],a],a],a],b],b],[b,a]]_{86}$,
$[[[[[[[[b,a],a],a],a],b],b],b],[b,a]]_{87}$, $[[[[[[[[b,a],a],a],b],b],b],b],[b,a]]_{88}$,
$[[[[[[[[b,a],a],b],b],b],b],b],[b,a]]_{89}$, $[[[[[[[[b,a],b],b],b],b],b],b],[b,a]]_{90}$,
$[[[[[[[[[b,a],a],a],a],a],a],a],a],a]_{91}$, $[[[[[[[[[b,a],a],a],a],a],a],a],a],b]_{92}$,
$[[[[[[[[[b,a],a],a],a],a],a],a],b],b]_{93}$, $[[[[[[[[[b,a],a],a],a],a],a],b],b],b]_{94}$,
$[[[[[[[[[b,a],a],a],a],a],b],b],b],b]_{95}$, $[[[[[[[[[b,a],a],a],a],b],b],b],b],b]_{96}$,
$[[[[[[[[[b,a],a],a],b],b],b],b],b],b]_{97}$, $[[[[[[[[[b,a],a],b],b],b],b],b],b],b]_{98}$,
$[[[[[[[[[b,a],b],b],b],b],b],b],b],b]_{99}$.

C.7 Degree 12

$[[[[[b,a],a],a],[b,a]],[[[b,a],b],[[b,a],a]]]_{1}$, $[[[[[b,a],a],b],[b,a]],[[[b,a],b],[[b,a],a]]]_{2}$,
$[[[[[b,a],a],b],[b,a]],[[[[b,a],a],a],[b,a]]]_{3}$, $[[[[[b,a],b],b],[b,a]],[[[b,a],b],[[b,a],a]]]_{4}$,
$[[[[[b,a],b],b],[b,a]],[[[[b,a],a],a],[b,a]]]_{5}$, $[[[[[b,a],b],b],[b,a]],[[[[b,a],a],b],[b,a]]]_{6}$,
$[[[[[[b,a],a],a],a],a],[[[b,a],b],[[b,a],a]]]_{7}$, $[[[[[[b,a],a],a],a],a],[[[[b,a],a],a],[b,a]]]_{8}$,
$[[[[[[b,a],a],a],a],a],[[[[b,a],a],b],[b,a]]]_{9}$, $[[[[[[b,a],a],a],a],a],[[[[b,a],b],b],[b,a]]]_{10}$,
$[[[[[[b,a],a],a],a],b],[[[b,a],b],[[b,a],a]]]_{11}$, $[[[[[[b,a],a],a],a],b],[[[[b,a],a],a],[b,a]]]_{12}$,
$[[[[[[b,a],a],a],a],b],[[[[b,a],a],b],[b,a]]]_{13}$, $[[[[[[b,a],a],a],a],b],[[[[b,a],b],b],[b,a]]]_{14}$,
$[[[[[[b,a],a],a],a],b],[[[[[b,a],a],a],a],a]]_{15}$, $[[[[[[b,a],a],a],b],b],[[[b,a],b],[[b,a],a]]]_{16}$,
$[[[[[[b,a],a],a],b],b],[[[[b,a],a],a],[b,a]]]_{17}$, $[[[[[[b,a],a],a],b],b],[[[[b,a],a],b],[b,a]]]_{18}$,
$[[[[[[b,a],a],a],b],b],[[[[b,a],b],b],[b,a]]]_{19}$, $[[[[[[b,a],a],a],b],b],[[[[[b,a],a],a],a],a]]_{20}$,
$[[[[[[b,a],a],a],b],b],[[[[[b,a],a],a],a],b]]_{21}$, $[[[[[[b,a],a],b],b],b],[[[b,a],b],[[b,a],a]]]_{22}$,
$[[[[[[b,a],a],b],b],b],[[[[b,a],a],a],[b,a]]]_{23}$, $[[[[[[b,a],a],b],b],b],[[[[b,a],a],b],[b,a]]]_{24}$,
$[[[[[[b,a],a],b],b],b],[[[[b,a],b],b],[b,a]]]_{25}$, $[[[[[[b,a],a],b],b],b],[[[[[b,a],a],a],a],a]]_{26}$,
$[[[[[[b,a],a],b],b],b],[[[[[b,a],a],a],a],b]]_{27}$, $[[[[[[b,a],a],b],b],b],[[[[[b,a],a],a],b],b]]_{28}$,
$[[[[[[b,a],b],b],b],b],[[[b,a],b],[[b,a],a]]]_{29}$, $[[[[[[b,a],b],b],b],b],[[[[b,a],a],a],[b,a]]]_{30}$,
$[[[[[[b,a],b],b],b],b],[[[[b,a],a],b],[b,a]]]_{31}$, $[[[[[[b,a],b],b],b],b],[[[[b,a],b],b],[b,a]]]_{32}$,
$[[[[[[b,a],b],b],b],b],[[[[[b,a],a],a],a],a]]_{33}$, $[[[[[[b,a],b],b],b],b],[[[[[b,a],a],a],a],b]]_{34}$,

$[[[[[[b,a],b],b],b],b],[[[[[b,a],a],a],b],b]]_{35}$, $[[[[[[b,a],b],b],b],b],[[[[[b,a],a],b],b],b]]_{36}$,

$[[[[[b,a],a],a],[[b,a],a]],[[[b,a],a],[b,a]]]_{37}$, $[[[[[b,a],a],a],[[b,a],a]],[[[b,a],b],[b,a]]]_{38}$,

$[[[[[b,a],a],a],[[b,a],a]],[[[[b,a],a],a],a]]_{39}$, $[[[[[b,a],a],a],[[b,a],a]],[[[[b,a],a],a],b]]_{40}$,

$[[[[[b,a],a],a],[[b,a],a]],[[[[b,a],a],b],b]]_{41}$, $[[[[[b,a],a],a],[[b,a],a]],[[[[b,a],b],b],b]]_{42}$,

$[[[[[b,a],a],a],[[b,a],b]],[[[b,a],a],[b,a]]]_{43}$, $[[[[[b,a],a],a],[[b,a],b]],[[[b,a],b],[b,a]]]_{44}$,

$[[[[[b,a],a],a],[[b,a],b]],[[[[b,a],a],a],a]]_{45}$, $[[[[[b,a],a],a],[[b,a],b]],[[[[b,a],a],a],b]]_{46}$,

$[[[[[b,a],a],a],[[b,a],b]],[[[[b,a],a],b],b]]_{47}$, $[[[[[b,a],a],a],[[b,a],b]],[[[[b,a],b],b],b]]_{48}$,

$[[[[[b,a],a],b],[[b,a],a]],[[[b,a],a],[b,a]]]_{49}$, $[[[[[b,a],a],b],[[b,a],a]],[[[b,a],b],[b,a]]]_{50}$,

$[[[[[b,a],a],b],[[b,a],a]],[[[[b,a],a],a],a]]_{51}$, $[[[[[b,a],a],b],[[b,a],a]],[[[[b,a],a],a],b]]_{52}$,

$[[[[[b,a],a],b],[[b,a],a]],[[[[b,a],a],b],b]]_{53}$, $[[[[[b,a],a],b],[[b,a],a]],[[[[b,a],b],b],b]]_{54}$,

$[[[[[b,a],a],b],[[b,a],b]],[[[b,a],a],[b,a]]]_{55}$, $[[[[[b,a],a],b],[[b,a],b]],[[[b,a],b],[b,a]]]_{56}$,

$[[[[[b,a],a],b],[[b,a],b]],[[[[b,a],a],a],a]]_{57}$, $[[[[[b,a],a],b],[[b,a],b]],[[[[b,a],a],a],b]]_{58}$,

$[[[[[b,a],a],b],[[b,a],b]],[[[[b,a],a],b],b]]_{59}$, $[[[[[b,a],a],b],[[b,a],b]],[[[[b,a],b],b],b]]_{60}$,

$[[[[[b,a],b],b],[[b,a],a]],[[[b,a],a],[b,a]]]_{61}$, $[[[[[b,a],b],b],[[b,a],a]],[[[b,a],b],[b,a]]]_{62}$,

$[[[[[b,a],b],b],[[b,a],a]],[[[[b,a],a],a],a]]_{63}$, $[[[[[b,a],b],b],[[b,a],a]],[[[[b,a],a],a],b]]_{64}$,

$[[[[[b,a],b],b],[[b,a],a]],[[[[b,a],a],b],b]]_{65}$, $[[[[[b,a],b],b],[[b,a],a]],[[[[b,a],b],b],b]]_{66}$,

$[[[[[b,a],b],b],[[b,a],b]],[[[b,a],a],[b,a]]]_{67}$, $[[[[[b,a],b],b],[[b,a],b]],[[[b,a],b],[b,a]]]_{68}$,

$[[[[[b,a],b],b],[[b,a],b]],[[[[b,a],a],a],a]]_{69}$, $[[[[[b,a],b],b],[[b,a],b]],[[[[b,a],a],a],b]]_{70}$,

$[[[[[b,a],b],b],[[b,a],b]],[[[[b,a],a],b],b]]_{71}$, $[[[[[b,a],b],b],[[b,a],b]],[[[[b,a],b],b],b]]_{72}$,

$[[[[[b,a],a],[b,a]],[b,a]],[[[b,a],a],[b,a]]]_{73}$, $[[[[[b,a],a],[b,a]],[b,a]],[[[b,a],b],[b,a]]]_{74}$,

$[[[[[b,a],a],[b,a]],[b,a]],[[[[b,a],a],a],a]]_{75}$, $[[[[[b,a],a],[b,a]],[b,a]],[[[[b,a],a],a],b]]_{76}$,

$[[[[[b,a],a],[b,a]],[b,a]],[[[[b,a],a],b],b]]_{77}$, $[[[[[b,a],a],[b,a]],[b,a]],[[[[b,a],b],b],b]]_{78}$,

$[[[[[b,a],b],[b,a]],[b,a]],[[[b,a],a],[b,a]]]_{79}$, $[[[[[b,a],b],[b,a]],[b,a]],[[[b,a],b],[b,a]]]_{80}$,

$[[[[[b,a],b],[b,a]],[b,a]],[[[[b,a],a],a],a]]_{81}$, $[[[[[b,a],b],[b,a]],[b,a]],[[[[b,a],a],a],b]]_{82}$,

$[[[[[b,a],b],[b,a]],[b,a]],[[[[b,a],a],b],b]]_{83}$, $[[[[[b,a],b],[b,a]],[b,a]],[[[[b,a],b],b],b]]_{84}$,

$[[[[[[b,a],a],a],a],[b,a]],[[[b,a],a],[b,a]]]_{85}$, $[[[[[[b,a],a],a],a],[b,a]],[[[b,a],b],[b,a]]]_{86}$,

$[[[[[[b,a],a],a],a],[b,a]],[[[[b,a],a],a],a]]_{87}$, $[[[[[[b,a],a],a],a],[b,a]],[[[[b,a],a],a],b]]_{88}$,

$[[[[[[b,a],a],a],a],[b,a]],[[[[b,a],a],b],b]]_{89}$, $[[[[[[b,a],a],a],a],[b,a]],[[[[b,a],b],b],b]]_{90}$,

$[[[[[[b,a],a],a],b],[b,a]],[[[b,a],a],[b,a]]]_{91}$, $[[[[[[b,a],a],a],b],[b,a]],[[[b,a],b],[b,a]]]_{92}$,

$[[[[[[b,a],a],a],b],[b,a]],[[[[b,a],a],a],a]]_{93}$, $[[[[[[b,a],a],a],b],[b,a]],[[[[b,a],a],a],b]]_{94}$,

$[[[[[[b,a],a],a],b],[b,a]],[[[[b,a],a],b],b]]_{95}$, $[[[[[[b,a],a],a],b],[b,a]],[[[[b,a],b],b],b]]_{96}$,

$[[[[[[b,a],a],b],b],[b,a]],[[[b,a],a],[b,a]]]_{97}$, $[[[[[[b,a],a],b],b],[b,a]],[[[b,a],b],[b,a]]]_{98}$,

$[[[[[[b,a],a],b],b],[b,a]],[[[[b,a],a],a],a]]_{99}$, $[[[[[[b,a],a],b],b],[b,a]],[[[[b,a],a],a],b]]_{100}$,

$[[[[[[b,a],a],b],b],[b,a]],[[[[b,a],a],b],b]]_{101}$, $[[[[[[b,a],a],b],b],[b,a]],[[[[b,a],b],b],b]]_{102}$,

$[[[[[[b,a],b],b],b],[b,a]],[[[b,a],a],[b,a]]]_{103}$, $[[[[[[b,a],b],b],b],[b,a]],[[[b,a],b],[b,a]]]_{104}$,

$[[[[[[b,a],b],b],b],[b,a]],[[[[b,a],a],a],a]]_{105}$, $[[[[[[b,a],b],b],b],[b,a]],[[[[b,a],a],a],b]]_{106}$,

$[[[[[[b,a],b],b],b],[b,a]],[[[[b,a],a],b],b]]_{107}$, $[[[[[[b,a],b],b],b],[b,a]],[[[[b,a],b],b],b]]_{108}$,

$[[[[[[b,a],a],a],a],a],a],[[[b,a],a],[b,a]]]_{109}$, $[[[[[[b,a],a],a],a],a],a],[[[b,a],b],[b,a]]]_{110}$,
$[[[[[[b,a],a],a],a],a],a],[[[[b,a],a],a],a]]_{111}$, $[[[[[[b,a],a],a],a],a],a],[[[[b,a],a],a],b]]_{112}$,
$[[[[[[b,a],a],a],a],a],a],[[[[b,a],a],b],b]]_{113}$, $[[[[[[b,a],a],a],a],a],a],[[[[b,a],b],b],b]]_{114}$,
$[[[[[[b,a],a],a],a],a],b],[[[b,a],a],[b,a]]]_{115}$, $[[[[[[b,a],a],a],a],a],b],[[[b,a],b],[b,a]]]_{116}$,
$[[[[[[b,a],a],a],a],a],b],[[[[b,a],a],a],a]]_{117}$, $[[[[[[b,a],a],a],a],a],b],[[[[b,a],a],a],b]]_{118}$,
$[[[[[[b,a],a],a],a],a],b],[[[[b,a],a],b],b]]_{119}$, $[[[[[[b,a],a],a],a],a],b],[[[[b,a],b],b],b]]_{120}$,
$[[[[[[b,a],a],a],a],b],b],[[[b,a],a],[b,a]]]_{121}$, $[[[[[[b,a],a],a],a],b],b],[[[b,a],b],[b,a]]]_{122}$,
$[[[[[[b,a],a],a],a],b],b],[[[[b,a],a],a],a]]_{123}$, $[[[[[[b,a],a],a],a],b],b],[[[[b,a],a],a],b]]_{124}$,
$[[[[[[b,a],a],a],a],b],b],[[[[b,a],a],b],b]]_{125}$, $[[[[[[b,a],a],a],a],b],b],[[[[b,a],b],b],b]]_{126}$,
$[[[[[[b,a],a],a],b],b],b],[[[b,a],a],[b,a]]]_{127}$, $[[[[[[b,a],a],a],b],b],b],[[[b,a],b],[b,a]]]_{128}$,
$[[[[[[b,a],a],a],b],b],b],[[[[b,a],a],a],a]]_{129}$, $[[[[[[b,a],a],a],b],b],b],[[[[b,a],a],a],b]]_{130}$,
$[[[[[[b,a],a],a],b],b],b],[[[[b,a],a],b],b]]_{131}$, $[[[[[[b,a],a],a],b],b],b],[[[[b,a],b],b],b]]_{132}$,
$[[[[[[b,a],a],b],b],b],b],[[[b,a],a],[b,a]]]_{133}$, $[[[[[[b,a],a],b],b],b],b],[[[b,a],b],[b,a]]]_{134}$,
$[[[[[[b,a],a],b],b],b],b],[[[[b,a],a],a],a]]_{135}$, $[[[[[[b,a],a],b],b],b],b],[[[[b,a],a],a],b]]_{136}$,
$[[[[[[b,a],a],b],b],b],b],[[[[b,a],a],b],b]]_{137}$, $[[[[[[b,a],a],b],b],b],b],[[[[b,a],b],b],b]]_{138}$,
$[[[[[[b,a],b],b],b],b],b],[[[b,a],a],[b,a]]]_{139}$, $[[[[[[b,a],b],b],b],b],b],[[[b,a],b],[b,a]]]_{140}$,
$[[[[[[b,a],b],b],b],b],b],[[[[b,a],a],a],a]]_{141}$, $[[[[[[b,a],b],b],b],b],b],[[[[b,a],a],a],b]]_{142}$,
$[[[[[[b,a],b],b],b],b],b],[[[[b,a],a],b],b]]_{143}$, $[[[[[[b,a],b],b],b],b],b],[[[[b,a],b],b],b]]_{144}$,
$[[[[b,a],a],b],[[[b,a],a],a]],[[[b,a],a],a]]_{145}$, $[[[[b,a],a],b],[[[b,a],a],a]],[[[b,a],a],b]]_{146}$,
$[[[[b,a],a],b],[[[b,a],a],a]],[[[b,a],b],b]]_{147}$, $[[[[b,a],b],b],[[[b,a],a],a]],[[[b,a],a],a]]_{148}$,
$[[[[b,a],b],b],[[[b,a],a],a]],[[[b,a],a],b]]_{149}$, $[[[[b,a],b],b],[[[b,a],a],a]],[[[b,a],b],b]]_{150}$,
$[[[[b,a],b],b],[[[b,a],a],b]],[[[b,a],a],b]]_{151}$, $[[[[b,a],b],b],[[[b,a],a],b]],[[[b,a],b],b]]_{152}$,
$[[[[b,a],a],[b,a]],[[b,a],a]],[[[b,a],a],a]]_{153}$, $[[[[b,a],a],[b,a]],[[b,a],a]],[[[b,a],a],b]]_{154}$,
$[[[[b,a],a],[b,a]],[[b,a],a]],[[[b,a],b],b]]_{155}$, $[[[[b,a],a],[b,a]],[[b,a],b]],[[[b,a],a],a]]_{156}$,
$[[[[b,a],a],[b,a]],[[b,a],b]],[[[b,a],a],b]]_{157}$, $[[[[b,a],a],[b,a]],[[b,a],b]],[[[b,a],b],b]]_{158}$,
$[[[[b,a],b],[b,a]],[[b,a],a]],[[[b,a],a],a]]_{159}$, $[[[[b,a],b],[b,a]],[[b,a],a]],[[[b,a],a],b]]_{160}$,
$[[[[b,a],b],[b,a]],[[b,a],a]],[[[b,a],b],b]]_{161}$, $[[[[b,a],b],[b,a]],[[b,a],b]],[[[b,a],a],a]]_{162}$,
$[[[[b,a],b],[b,a]],[[b,a],b]],[[[b,a],a],b]]_{163}$, $[[[[b,a],b],[b,a]],[[b,a],b]],[[[b,a],b],b]]_{164}$,
$[[[[[b,a],a],a],a],[[b,a],a]],[[[b,a],a],a]]_{165}$, $[[[[[b,a],a],a],a],[[b,a],a]],[[[b,a],a],b]]_{166}$,
$[[[[[b,a],a],a],a],[[b,a],a]],[[[b,a],b],b]]_{167}$, $[[[[[b,a],a],a],a],[[b,a],b]],[[[b,a],a],a]]_{168}$,
$[[[[[b,a],a],a],a],[[b,a],b]],[[[b,a],a],b]]_{169}$, $[[[[[b,a],a],a],a],[[b,a],b]],[[[b,a],b],b]]_{170}$,
$[[[[[b,a],a],a],b],[[b,a],a]],[[[b,a],a],a]]_{171}$, $[[[[[b,a],a],a],b],[[b,a],a]],[[[b,a],a],b]]_{172}$,
$[[[[[b,a],a],a],b],[[b,a],a]],[[[b,a],b],b]]_{173}$, $[[[[[b,a],a],a],b],[[b,a],b]],[[[b,a],a],a]]_{174}$,
$[[[[[b,a],a],a],b],[[b,a],b]],[[[b,a],a],b]]_{175}$, $[[[[[b,a],a],a],b],[[b,a],b]],[[[b,a],b],b]]_{176}$,
$[[[[[b,a],a],b],b],[[b,a],a]],[[[b,a],a],a]]_{177}$, $[[[[[b,a],a],b],b],[[b,a],a]],[[[b,a],a],b]]_{178}$,
$[[[[[b,a],a],b],b],[[b,a],a]],[[[b,a],b],b]]_{179}$, $[[[[[b,a],a],b],b],[[b,a],b]],[[[b,a],a],a]]_{180}$,
$[[[[[b,a],a],b],b],[[b,a],b]],[[[b,a],a],b]]_{181}$, $[[[[[b,a],a],b],b],[[b,a],b]],[[[b,a],b],b]]_{182}$,

$[[[[[b,a],b],b],b],[[b,a],a]],[[[b,a],a],a]]_{183}$, $[[[[[b,a],b],b],b],[[b,a],a]],[[[b,a],a],b]]_{184}$,
$[[[[[b,a],b],b],b],[[b,a],a]],[[[b,a],b],b]]_{185}$, $[[[[[b,a],b],b],b],[[b,a],b]],[[[b,a],a],a]]_{186}$,
$[[[[[b,a],b],b],b],[[b,a],b]],[[[b,a],a],b]]_{187}$, $[[[[[b,a],b],b],b],[[b,a],b]],[[[b,a],b],b]]_{188}$,
$[[[[[b,a],a],a],[b,a]],[b,a]],[[[b,a],a],a]]_{189}$, $[[[[[b,a],a],a],[b,a]],[b,a]],[[[b,a],a],b]]_{190}$,
$[[[[[b,a],a],a],[b,a]],[b,a]],[[[b,a],b],b]]_{191}$, $[[[[[b,a],a],b],[b,a]],[b,a]],[[[b,a],a],a]]_{192}$,
$[[[[[b,a],a],b],[b,a]],[b,a]],[[[b,a],a],b]]_{193}$, $[[[[[b,a],a],b],[b,a]],[b,a]],[[[b,a],b],b]]_{194}$,
$[[[[[b,a],b],b],[b,a]],[b,a]],[[[b,a],a],a]]_{195}$, $[[[[[b,a],b],b],[b,a]],[b,a]],[[[b,a],a],b]]_{196}$,
$[[[[[b,a],b],b],[b,a]],[b,a]],[[[b,a],b],b]]_{197}$, $[[[[[[b,a],a],a],a],a],[b,a]],[[[b,a],a],a]]_{198}$,
$[[[[[[b,a],a],a],a],a],[b,a]],[[[b,a],a],b]]_{199}$, $[[[[[[b,a],a],a],a],a],[b,a]],[[[b,a],b],b]]_{200}$,
$[[[[[[b,a],a],a],a],b],[b,a]],[[[b,a],a],a]]_{201}$, $[[[[[[b,a],a],a],a],b],[b,a]],[[[b,a],a],b]]_{202}$,
$[[[[[[b,a],a],a],a],b],[b,a]],[[[b,a],b],b]]_{203}$, $[[[[[[b,a],a],a],b],b],[b,a]],[[[b,a],a],a]]_{204}$,
$[[[[[[b,a],a],a],b],b],[b,a]],[[[b,a],a],b]]_{205}$, $[[[[[[b,a],a],a],b],b],[b,a]],[[[b,a],b],b]]_{206}$,
$[[[[[[b,a],a],b],b],b],[b,a]],[[[b,a],a],a]]_{207}$, $[[[[[[b,a],a],b],b],b],[b,a]],[[[b,a],a],b]]_{208}$,
$[[[[[[b,a],a],b],b],b],[b,a]],[[[b,a],b],b]]_{209}$, $[[[[[[b,a],b],b],b],b],[b,a]],[[[b,a],a],a]]_{210}$,
$[[[[[[b,a],b],b],b],b],[b,a]],[[[b,a],a],b]]_{211}$, $[[[[[[b,a],b],b],b],b],[b,a]],[[[b,a],b],b]]_{212}$,
$[[[[[[[b,a],a],a],a],a],a],a],[[[b,a],a],a]]_{213}$, $[[[[[[[b,a],a],a],a],a],a],a],[[[b,a],a],b]]_{214}$,
$[[[[[[[b,a],a],a],a],a],a],a],[[[b,a],b],b]]_{215}$, $[[[[[[[b,a],a],a],a],a],a],b],[[[b,a],a],a]]_{216}$,
$[[[[[[[b,a],a],a],a],a],a],b],[[[b,a],a],b]]_{217}$, $[[[[[[[b,a],a],a],a],a],a],b],[[[b,a],b],b]]_{218}$,
$[[[[[[[b,a],a],a],a],a],b],b],[[[b,a],a],a]]_{219}$, $[[[[[[[b,a],a],a],a],a],b],b],[[[b,a],a],b]]_{220}$,
$[[[[[[[b,a],a],a],a],a],b],b],[[[b,a],b],b]]_{221}$, $[[[[[[[b,a],a],a],a],b],b],b],[[[b,a],a],a]]_{222}$,
$[[[[[[[b,a],a],a],a],b],b],b],[[[b,a],a],b]]_{223}$, $[[[[[[[b,a],a],a],a],b],b],b],[[[b,a],b],b]]_{224}$,
$[[[[[[[b,a],a],a],b],b],b],b],[[[b,a],a],a]]_{225}$, $[[[[[[[b,a],a],a],b],b],b],b],[[[b,a],a],b]]_{226}$,
$[[[[[[[b,a],a],a],b],b],b],b],[[[b,a],b],b]]_{227}$, $[[[[[[[b,a],a],b],b],b],b],b],[[[b,a],a],a]]_{228}$,
$[[[[[[[b,a],a],b],b],b],b],b],[[[b,a],a],b]]_{229}$, $[[[[[[[b,a],a],b],b],b],b],b],[[[b,a],b],b]]_{230}$,
$[[[[[[[b,a],b],b],b],b],b],b],[[[b,a],a],a]]_{231}$, $[[[[[[[b,a],b],b],b],b],b],b],[[[b,a],a],b]]_{232}$,
$[[[[[[[b,a],b],b],b],b],b],b],[[[b,a],b],b]]_{233}$, $[[[[[b,a],b],[[b,a],a]],[[b,a],a]],[[b,a],a]]_{234}$,
$[[[[[b,a],b],[[b,a],a]],[[b,a],a]],[[b,a],b]]_{235}$, $[[[[[b,a],b],[[b,a],a]],[[b,a],b]],[[b,a],b]]_{236}$,
$[[[[[b,a],a],a],[b,a]],[[b,a],a]],[[b,a],a]]_{237}$, $[[[[[b,a],a],a],[b,a]],[[b,a],a]],[[b,a],b]]_{238}$,
$[[[[[b,a],a],a],[b,a]],[[b,a],b]],[[b,a],b]]_{239}$, $[[[[[b,a],a],b],[b,a]],[[b,a],a]],[[b,a],a]]_{240}$,
$[[[[[b,a],a],b],[b,a]],[[b,a],a]],[[b,a],b]]_{241}$, $[[[[[b,a],a],b],[b,a]],[[b,a],b]],[[b,a],b]]_{242}$,
$[[[[[b,a],b],b],[b,a]],[[b,a],a]],[[b,a],a]]_{243}$, $[[[[[b,a],b],b],[b,a]],[[b,a],a]],[[b,a],b]]_{244}$,
$[[[[[b,a],b],b],[b,a]],[[b,a],b]],[[b,a],b]]_{245}$, $[[[[[[b,a],a],a],a],a],[[b,a],a]],[[b,a],a]]_{246}$,
$[[[[[[b,a],a],a],a],a],[[b,a],a]],[[b,a],b]]_{247}$, $[[[[[[b,a],a],a],a],a],[[b,a],b]],[[b,a],b]]_{248}$,
$[[[[[[b,a],a],a],a],b],[[b,a],a]],[[b,a],a]]_{249}$, $[[[[[[b,a],a],a],a],b],[[b,a],a]],[[b,a],b]]_{250}$,
$[[[[[[b,a],a],a],a],b],[[b,a],b]],[[b,a],b]]_{251}$, $[[[[[[b,a],a],a],b],b],[[b,a],a]],[[b,a],a]]_{252}$,
$[[[[[[b,a],a],a],b],b],[[b,a],a]],[[b,a],b]]_{253}$, $[[[[[[b,a],a],a],b],b],[[b,a],b]],[[b,a],b]]_{254}$,
$[[[[[[b,a],a],b],b],b],[[b,a],a]],[[b,a],a]]_{255}$, $[[[[[[b,a],a],b],b],b],[[b,a],a]],[[b,a],b]]_{256}$,

$[[[[[[[b,a],a],b],b],b],[[b,a],b]],[[b,a],b]]_{257}$, $[[[[[[[b,a],b],b],b],b],[[b,a],a]],[[b,a],a]]_{258}$,
$[[[[[[[b,a],b],b],b],b],[[b,a],a]],[[b,a],b]]_{259}$, $[[[[[[[b,a],b],b],b],b],[[b,a],b]],[[b,a],b]]_{260}$,
$[[[[[[b,a],a],[b,a]],[b,a]],[b,a]],[[b,a],a]]_{261}$, $[[[[[[b,a],a],[b,a]],[b,a]],[b,a]],[[b,a],b]]_{262}$,
$[[[[[[b,a],b],[b,a]],[b,a]],[b,a]],[[b,a],a]]_{263}$, $[[[[[[b,a],b],[b,a]],[b,a]],[b,a]],[[b,a],b]]_{264}$,
$[[[[[[[b,a],a],a],a],[b,a]],[b,a]],[[b,a],a]]_{265}$, $[[[[[[[b,a],a],a],a],[b,a]],[b,a]],[[b,a],b]]_{266}$,
$[[[[[[[b,a],a],a],b],[b,a]],[b,a]],[[b,a],a]]_{267}$, $[[[[[[[b,a],a],a],b],[b,a]],[b,a]],[[b,a],b]]_{268}$,
$[[[[[[[b,a],a],b],b],[b,a]],[b,a]],[[b,a],a]]_{269}$, $[[[[[[[b,a],a],b],b],[b,a]],[b,a]],[[b,a],b]]_{270}$,
$[[[[[[[b,a],b],b],b],[b,a]],[b,a]],[[b,a],a]]_{271}$, $[[[[[[[b,a],b],b],b],[b,a]],[b,a]],[[b,a],b]]_{272}$,
$[[[[[[[[b,a],a],a],a],a],a],[b,a]],[[b,a],a]]_{273}$, $[[[[[[[[b,a],a],a],a],a],a],[b,a]],[[b,a],b]]_{274}$,
$[[[[[[[[b,a],a],a],a],a],b],[b,a]],[[b,a],a]]_{275}$, $[[[[[[[[b,a],a],a],a],a],b],[b,a]],[[b,a],b]]_{276}$,
$[[[[[[[[b,a],a],a],a],b],b],[b,a]],[[b,a],a]]_{277}$, $[[[[[[[[b,a],a],a],a],b],b],[b,a]],[[b,a],b]]_{278}$,
$[[[[[[[[b,a],a],a],b],b],b],[b,a]],[[b,a],a]]_{279}$, $[[[[[[[[b,a],a],a],b],b],b],[b,a]],[[b,a],b]]_{280}$,
$[[[[[[[[b,a],a],b],b],b],b],[b,a]],[[b,a],a]]_{281}$, $[[[[[[[[b,a],a],b],b],b],b],[b,a]],[[b,a],b]]_{282}$,
$[[[[[[[[b,a],b],b],b],b],b],[b,a]],[[b,a],a]]_{283}$, $[[[[[[[[b,a],b],b],b],b],b],[b,a]],[[b,a],b]]_{284}$,
$[[[[[[[[[b,a],a],a],a],a],a],a],a],[[b,a],a]]_{285}$, $[[[[[[[[[b,a],a],a],a],a],a],a],a],[[b,a],b]]_{286}$,
$[[[[[[[[[b,a],a],a],a],a],a],a],b],[[b,a],a]]_{287}$, $[[[[[[[[[b,a],a],a],a],a],a],a],b],[[b,a],b]]_{288}$,
$[[[[[[[[[b,a],a],a],a],a],a],b],b],[[b,a],a]]_{289}$, $[[[[[[[[[b,a],a],a],a],a],a],b],b],[[b,a],b]]_{290}$,
$[[[[[[[[[b,a],a],a],a],a],b],b],b],[[b,a],a]]_{291}$, $[[[[[[[[[b,a],a],a],a],a],b],b],b],[[b,a],b]]_{292}$,
$[[[[[[[[[b,a],a],a],a],b],b],b],b],[[b,a],a]]_{293}$, $[[[[[[[[[b,a],a],a],a],b],b],b],b],[[b,a],b]]_{294}$,
$[[[[[[[[[b,a],a],a],b],b],b],b],b],[[b,a],a]]_{295}$, $[[[[[[[[[b,a],a],a],b],b],b],b],b],[[b,a],b]]_{296}$,
$[[[[[[[[[b,a],a],b],b],b],b],b],b],[[b,a],a]]_{297}$, $[[[[[[[[[b,a],a],b],b],b],b],b],b],[[b,a],b]]_{298}$,
$[[[[[[[[[b,a],b],b],b],b],b],b],b],[[b,a],a]]_{299}$, $[[[[[[[[[b,a],b],b],b],b],b],b],b],[[b,a],b]]_{300}$,
$[[[[[[[b,a],a],a],[b,a]],[b,a]],[b,a]],[b,a]]_{301}$, $[[[[[[[b,a],a],b],[b,a]],[b,a]],[b,a]],[b,a]]_{302}$,
$[[[[[[[b,a],b],b],[b,a]],[b,a]],[b,a]],[b,a]]_{303}$, $[[[[[[[[b,a],a],a],a],a],[b,a]],[b,a]],[b,a]]_{304}$,
$[[[[[[[[b,a],a],a],a],b],[b,a]],[b,a]],[b,a]]_{305}$, $[[[[[[[[b,a],a],a],b],b],[b,a]],[b,a]],[b,a]]_{306}$,
$[[[[[[[[b,a],a],b],b],b],[b,a]],[b,a]],[b,a]]_{307}$, $[[[[[[[[b,a],b],b],b],b],[b,a]],[b,a]],[b,a]]_{308}$,
$[[[[[[[[[b,a],a],a],a],a],a],a],[b,a]],[b,a]]_{309}$, $[[[[[[[[[b,a],a],a],a],a],a],b],[b,a]],[b,a]]_{310}$,
$[[[[[[[[[b,a],a],a],a],a],b],b],[b,a]],[b,a]]_{311}$, $[[[[[[[[[b,a],a],a],a],b],b],b],[b,a]],[b,a]]_{312}$,
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$[[[[[[[[[b,a],b],b],b],b],b],b],[b,a]],[b,a]]_{315}$, $[[[[[[[[[[b,a],a],a],a],a],a],a],a],a],[b,a]]_{316}$,
$[[[[[[[[[[b,a],a],a],a],a],a],a],a],b],[b,a]]_{317}$, $[[[[[[[[[[b,a],a],a],a],a],a],a],b],b],[b,a]]_{318}$,
$[[[[[[[[[[b,a],a],a],a],a],a],b],b],b],[b,a]]_{319}$, $[[[[[[[[[[b,a],a],a],a],a],b],b],b],b],[b,a]]_{320}$,
$[[[[[[[[[[b,a],a],a],a],b],b],b],b],b],[b,a]]_{321}$, $[[[[[[[[[[b,a],a],a],b],b],b],b],b],b],[b,a]]_{322}$,
$[[[[[[[[[[b,a],a],b],b],b],b],b],b],b],[b,a]]_{323}$, $[[[[[[[[[[b,a],b],b],b],b],b],b],b],b],[b,a]]_{324}$,
$[[[[[[[[[[b,a],a],a],a],a],a],a],a],a],a],a]_{325}$, $[[[[[[[[[[b,a],a],a],a],a],a],a],a],a],a],b]_{326}$,
$[[[[[[[[[[b,a],a],a],a],a],a],a],a],a],b],b]_{327}$, $[[[[[[[[[[b,a],a],a],a],a],a],a],a],b],b],b]_{328}$,
$[[[[[[[[[[b,a],a],a],a],a],a],a],b],b],b],b]_{329}$, $[[[[[[[[[[b,a],a],a],a],a],a],b],b],b],b],b]_{330}$,

$[[[[[[[[[[b, a], a], a], a], a], b], b], b], b], b]_{331}$, $[[[[[[[[[[b, a], a], a], a], b], b], b], b], b], b]_{332}$,
$[[[[[[[[[[b, a], a], a], b], b], b], b], b], b], b]_{333}$, $[[[[[[[[[[b, a], a], b], b], b], b], b], b], b], b]_{334}$,
$[[[[[[[[[[b, a], b], b], b], b], b], b], b], b], b]_{335}$.

Appendix D

Hall words of 3 generators

D.1 Degree 5

$$\begin{array}{lll}
[[[b,a],a],[b,a]]_1, & [[[b,a],a],[c,a]]_2, & [[[b,a],a],[c,b]]_3, \\
[[[b,a],b],[b,a]]_4, & [[[b,a],b],[c,a]]_5, & [[[b,a],b],[c,b]]_6, \\
[[[b,a],c],[b,a]]_7, & [[[b,a],c],[c,a]]_8, & [[[b,a],c],[c,b]]_9, \\
[[[c,a],a],[b,a]]_{10}, & [[[c,a],a],[c,a]]_{11}, & [[[c,a],a],[c,b]]_{12}, \\
[[[c,a],b],[b,a]]_{13}, & [[[c,a],b],[c,a]]_{14}, & [[[c,a],b],[c,b]]_{15}, \\
[[[c,a],c],[b,a]]_{16}, & [[[c,a],c],[c,a]]_{17}, & [[[c,a],c],[c,b]]_{18}, \\
[[[c,b],b],[b,a]]_{19}, & [[[c,b],b],[c,a]]_{20}, & [[[c,b],b],[c,b]]_{21}, \\
[[[c,b],c],[b,a]]_{22}, & [[[c,b],c],[c,a]]_{23}, & [[[c,b],c],[c,b]]_{24}, \\
[[[[b,a],a],a],a]_{25}, & [[[[b,a],a],a],b]_{26}, & [[[[b,a],a],a],c]_{27}, \\
[[[[b,a],a],b],b]_{28}, & [[[[b,a],a],b],c]_{29}, & [[[[b,a],a],c],c]_{30}, \\
[[[[b,a],b],b],b]_{31}, & [[[[b,a],b],b],c]_{32}, & [[[[b,a],b],c],c]_{33}, \\
[[[[b,a],c],c],c]_{34}, & [[[[c,a],a],a],a]_{35}, & [[[[c,a],a],a],b]_{36}, \\
[[[[c,a],a],a],c]_{37}, & [[[[c,a],a],b],b]_{38}, & [[[[c,a],a],b],c]_{39}, \\
[[[[c,a],a],c],c]_{40}, & [[[[c,a],b],b],b]_{41}, & [[[[c,a],b],b],c]_{42}, \\
[[[[c,a],b],c],c]_{43}, & [[[[c,a],c],c],c]_{44}, & [[[[c,b],b],b],b]_{45}, \\
[[[[c,b],b],b],c]_{46}, & [[[[c,b],b],c],c]_{47}, & [[[[c,b],c],c],c]_{48}.
\end{array}$$

D.2 Degree 6

$$\begin{array}{lll}
[[[b,a],b],[[b,a],a]]_1, & [[[b,a],c],[[b,a],a]]_2, & [[[b,a],c],[[b,a],b]]_3, \\
[[[c,a],a],[[b,a],a]]_4, & [[[c,a],a],[[b,a],b]]_5, & [[[c,a],a],[[b,a],c]]_6, \\
[[[c,a],b],[[b,a],a]]_7, & [[[c,a],b],[[b,a],b]]_8, & [[[c,a],b],[[b,a],c]]_9, \\
[[[c,a],b],[[c,a],a]]_{10}, & [[[c,a],c],[[b,a],a]]_{11}, & [[[c,a],c],[[b,a],b]]_{12}, \\
[[[c,a],c],[[b,a],c]]_{13}, & [[[c,a],c],[[c,a],a]]_{14}, & [[[c,a],c],[[c,a],b]]_{15}, \\
[[[c,b],b],[[b,a],a]]_{16}, & [[[c,b],b],[[b,a],b]]_{17}, & [[[c,b],b],[[b,a],c]]_{18}, \\
[[[c,b],b],[[c,a],a]]_{19}, & [[[c,b],b],[[c,a],b]]_{20}, & [[[c,b],b],[[c,a],c]]_{21}, \\
[[[c,b],c],[[b,a],a]]_{22}, & [[[c,b],c],[[b,a],b]]_{23}, & [[[c,b],c],[[b,a],c]]_{24}, \\
[[[c,b],c],[[c,a],a]]_{25}, & [[[c,b],c],[[c,a],b]]_{26}, & [[[c,b],c],[[c,a],c]]_{27}, \\
[[[c,b],c],[[c,b],b]]_{28}, & [[[c,a],[b,a]],[b,a]]_{29}, & [[[c,a],[b,a]],[c,a]]_{30},
\end{array}$$

$[[[c,a],[b,a]],[c,b]]_{31}$, $[[[c,b],[b,a]],[b,a]]_{32}$, $[[[c,b],[b,a]],[c,a]]_{33}$,
$[[[c,b],[b,a]],[c,b]]_{34}$, $[[[c,b],[c,a]],[c,a]]_{35}$, $[[[c,b],[c,a]],[c,b]]_{36}$,
$[[[[b,a],a],a],[b,a]]_{37}$, $[[[[b,a],a],a],[c,a]]_{38}$, $[[[[b,a],a],a],[c,b]]_{39}$,
$[[[[b,a],a],b],[b,a]]_{40}$, $[[[[b,a],a],b],[c,a]]_{41}$, $[[[[b,a],a],b],[c,b]]_{42}$,
$[[[[b,a],a],c],[b,a]]_{43}$, $[[[[b,a],a],c],[c,a]]_{44}$, $[[[[b,a],a],c],[c,b]]_{45}$,
$[[[[b,a],b],b],[b,a]]_{46}$, $[[[[b,a],b],b],[c,a]]_{47}$, $[[[[b,a],b],b],[c,b]]_{48}$,
$[[[[b,a],b],c],[b,a]]_{49}$, $[[[[b,a],b],c],[c,a]]_{50}$, $[[[[b,a],b],c],[c,b]]_{51}$,
$[[[[b,a],c],c],[b,a]]_{52}$, $[[[[b,a],c],c],[c,a]]_{53}$, $[[[[b,a],c],c],[c,b]]_{54}$,
$[[[[c,a],a],a],[b,a]]_{55}$, $[[[[c,a],a],a],[c,a]]_{56}$, $[[[[c,a],a],a],[c,b]]_{57}$,
$[[[[c,a],a],b],[b,a]]_{58}$, $[[[[c,a],a],b],[c,a]]_{59}$, $[[[[c,a],a],b],[c,b]]_{60}$,
$[[[[c,a],a],c],[b,a]]_{61}$, $[[[[c,a],a],c],[c,a]]_{62}$, $[[[[c,a],a],c],[c,b]]_{63}$,
$[[[[c,a],b],b],[b,a]]_{64}$, $[[[[c,a],b],b],[c,a]]_{65}$, $[[[[c,a],b],b],[c,b]]_{66}$,
$[[[[c,a],b],c],[b,a]]_{67}$, $[[[[c,a],b],c],[c,a]]_{68}$, $[[[[c,a],b],c],[c,b]]_{69}$,
$[[[[c,a],c],c],[b,a]]_{70}$, $[[[[c,a],c],c],[c,a]]_{71}$, $[[[[c,a],c],c],[c,b]]_{72}$,
$[[[[c,b],b],b],[b,a]]_{73}$, $[[[[c,b],b],b],[c,a]]_{74}$, $[[[[c,b],b],b],[c,b]]_{75}$,
$[[[[c,b],b],c],[b,a]]_{76}$, $[[[[c,b],b],c],[c,a]]_{77}$, $[[[[c,b],b],c],[c,b]]_{78}$,
$[[[[c,b],c],c],[b,a]]_{79}$, $[[[[c,b],c],c],[c,a]]_{80}$, $[[[[c,b],c],c],[c,b]]_{81}$,
$[[[[[b,a],a],a],a],a]_{82}$, $[[[[[b,a],a],a],a],b]_{83}$, $[[[[[b,a],a],a],a],c]_{84}$,
$[[[[[b,a],a],a],b],b]_{85}$, $[[[[[b,a],a],a],b],c]_{86}$, $[[[[[b,a],a],a],c],c]_{87}$,
$[[[[[b,a],a],b],b],b]_{88}$, $[[[[[b,a],a],b],b],c]_{89}$, $[[[[[b,a],a],b],c],c]_{90}$,
$[[[[[b,a],a],c],c],c]_{91}$, $[[[[[b,a],b],b],b],b]_{92}$, $[[[[[b,a],b],b],b],c]_{93}$,
$[[[[[b,a],b],b],c],c]_{94}$, $[[[[[b,a],b],c],c],c]_{95}$, $[[[[[b,a],c],c],c],c]_{96}$,
$[[[[[c,a],a],a],a],a]_{97}$, $[[[[[c,a],a],a],a],b]_{98}$, $[[[[[c,a],a],a],a],c]_{99}$,
$[[[[[c,a],a],a],b],b]_{100}$, $[[[[[c,a],a],a],b],c]_{101}$, $[[[[[c,a],a],a],c],c]_{102}$,
$[[[[[c,a],a],b],b],b]_{103}$, $[[[[[c,a],a],b],b],c]_{104}$, $[[[[[c,a],a],b],c],c]_{105}$,
$[[[[[c,a],a],c],c],c]_{106}$, $[[[[[c,a],b],b],b],b]_{107}$, $[[[[[c,a],b],b],b],c]_{108}$,
$[[[[[c,a],b],b],c],c]_{109}$, $[[[[[c,a],b],c],c],c]_{110}$, $[[[[[c,a],c],c],c],c]_{111}$,
$[[[[[c,b],b],b],b],b]_{112}$, $[[[[[c,b],b],b],b],c]_{113}$, $[[[[[c,b],b],b],c],c]_{114}$,
$[[[[[c,b],b],c],c],c]_{115}$, $[[[[[c,b],c],c],c],c]_{116}$.

D.3 Degree 7

$[[[c,a],[b,a]],[[b,a],a]]_1$, $[[[c,a],[b,a]],[[b,a],b]]_2$, $[[[c,a],[b,a]],[[b,a],c]]_3$,
$[[[c,a],[b,a]],[[c,a],a]]_4$, $[[[c,a],[b,a]],[[c,a],b]]_5$, $[[[c,a],[b,a]],[[c,a],c]]_6$,
$[[[c,a],[b,a]],[[c,b],b]]_7$, $[[[c,a],[b,a]],[[c,b],c]]_8$, $[[[c,b],[b,a]],[[b,a],a]]_9$,
$[[[c,b],[b,a]],[[b,a],b]]_{10}$, $[[[c,b],[b,a]],[[b,a],c]]_{11}$, $[[[c,b],[b,a]],[[c,a],a]]_{12}$,
$[[[c,b],[b,a]],[[c,a],b]]_{13}$, $[[[c,b],[b,a]],[[c,a],c]]_{14}$, $[[[c,b],[b,a]],[[c,b],b]]_{15}$,

$[[[c,b],[b,a]],[[c,b],c]]_{16}$, $[[[c,b],[c,a]],[[b,a],a]]_{17}$, $[[[c,b],[c,a]],[[b,a],b]]_{18}$,
$[[[c,b],[c,a]],[[b,a],c]]_{19}$, $[[[c,b],[c,a]],[[c,a],a]]_{20}$, $[[[c,b],[c,a]],[[c,a],b]]_{21}$,
$[[[c,b],[c,a]],[[c,a],c]]_{22}$, $[[[c,b],[c,a]],[[c,b],b]]_{23}$, $[[[c,b],[c,a]],[[c,b],c]]_{24}$,
$[[[[b,a],a],a],[[b,a],a]]_{25}$, $[[[[b,a],a],a],[[b,a],b]]_{26}$, $[[[[b,a],a],a],[[b,a],c]]_{27}$,
$[[[[b,a],a],a],[[c,a],a]]_{28}$, $[[[[b,a],a],a],[[c,a],b]]_{29}$, $[[[[b,a],a],a],[[c,a],c]]_{30}$,
$[[[[b,a],a],a],[[c,b],b]]_{31}$, $[[[[b,a],a],a],[[c,b],c]]_{32}$, $[[[[b,a],a],b],[[b,a],a]]_{33}$,
$[[[[b,a],a],b],[[b,a],b]]_{34}$, $[[[[b,a],a],b],[[b,a],c]]_{35}$, $[[[[b,a],a],b],[[c,a],a]]_{36}$,
$[[[[b,a],a],b],[[c,a],b]]_{37}$, $[[[[b,a],a],b],[[c,a],c]]_{38}$, $[[[[b,a],a],b],[[c,b],b]]_{39}$,
$[[[[b,a],a],b],[[c,b],c]]_{40}$, $[[[[b,a],a],c],[[b,a],a]]_{41}$, $[[[[b,a],a],c],[[b,a],b]]_{42}$,
$[[[[b,a],a],c],[[b,a],c]]_{43}$, $[[[[b,a],a],c],[[c,a],a]]_{44}$, $[[[[b,a],a],c],[[c,a],b]]_{45}$,
$[[[[b,a],a],c],[[c,a],c]]_{46}$, $[[[[b,a],a],c],[[c,b],b]]_{47}$, $[[[[b,a],a],c],[[c,b],c]]_{48}$,
$[[[[b,a],b],b],[[b,a],a]]_{49}$, $[[[[b,a],b],b],[[b,a],b]]_{50}$, $[[[[b,a],b],b],[[b,a],c]]_{51}$,
$[[[[b,a],b],b],[[c,a],a]]_{52}$, $[[[[b,a],b],b],[[c,a],b]]_{53}$, $[[[[b,a],b],b],[[c,a],c]]_{54}$,
$[[[[b,a],b],b],[[c,b],b]]_{55}$, $[[[[b,a],b],b],[[c,b],c]]_{56}$, $[[[[b,a],b],c],[[b,a],a]]_{57}$,
$[[[[b,a],b],c],[[b,a],b]]_{58}$, $[[[[b,a],b],c],[[b,a],c]]_{59}$, $[[[[b,a],b],c],[[c,a],a]]_{60}$,
$[[[[b,a],b],c],[[c,a],b]]_{61}$, $[[[[b,a],b],c],[[c,a],c]]_{62}$, $[[[[b,a],b],c],[[c,b],b]]_{63}$,
$[[[[b,a],b],c],[[c,b],c]]_{64}$, $[[[[b,a],c],c],[[b,a],a]]_{65}$, $[[[[b,a],c],c],[[b,a],b]]_{66}$,
$[[[[b,a],c],c],[[b,a],c]]_{67}$, $[[[[b,a],c],c],[[c,a],a]]_{68}$, $[[[[b,a],c],c],[[c,a],b]]_{69}$,
$[[[[b,a],c],c],[[c,a],c]]_{70}$, $[[[[b,a],c],c],[[c,b],b]]_{71}$, $[[[[b,a],c],c],[[c,b],c]]_{72}$,
$[[[[c,a],a],a],[[b,a],a]]_{73}$, $[[[[c,a],a],a],[[b,a],b]]_{74}$, $[[[[c,a],a],a],[[b,a],c]]_{75}$,
$[[[[c,a],a],a],[[c,a],a]]_{76}$, $[[[[c,a],a],a],[[c,a],b]]_{77}$, $[[[[c,a],a],a],[[c,a],c]]_{78}$,
$[[[[c,a],a],a],[[c,b],b]]_{79}$, $[[[[c,a],a],a],[[c,b],c]]_{80}$, $[[[[c,a],a],b],[[b,a],a]]_{81}$,
$[[[[c,a],a],b],[[b,a],b]]_{82}$, $[[[[c,a],a],b],[[b,a],c]]_{83}$, $[[[[c,a],a],b],[[c,a],a]]_{84}$,
$[[[[c,a],a],b],[[c,a],b]]_{85}$, $[[[[c,a],a],b],[[c,a],c]]_{86}$, $[[[[c,a],a],b],[[c,b],b]]_{87}$,
$[[[[c,a],a],b],[[c,b],c]]_{88}$, $[[[[c,a],a],c],[[b,a],a]]_{89}$, $[[[[c,a],a],c],[[b,a],b]]_{90}$,
$[[[[c,a],a],c],[[b,a],c]]_{91}$, $[[[[c,a],a],c],[[c,a],a]]_{92}$, $[[[[c,a],a],c],[[c,a],b]]_{93}$,
$[[[[c,a],a],c],[[c,a],c]]_{94}$, $[[[[c,a],a],c],[[c,b],b]]_{95}$, $[[[[c,a],a],c],[[c,b],c]]_{96}$,
$[[[[c,a],b],b],[[b,a],a]]_{97}$, $[[[[c,a],b],b],[[b,a],b]]_{98}$, $[[[[c,a],b],b],[[b,a],c]]_{99}$,
$[[[[c,a],b],b],[[c,a],a]]_{100}$, $[[[[c,a],b],b],[[c,a],b]]_{101}$, $[[[[c,a],b],b],[[c,a],c]]_{102}$,
$[[[[c,a],b],b],[[c,b],b]]_{103}$, $[[[[c,a],b],b],[[c,b],c]]_{104}$, $[[[[c,a],b],c],[[b,a],a]]_{105}$,
$[[[[c,a],b],c],[[b,a],b]]_{106}$, $[[[[c,a],b],c],[[b,a],c]]_{107}$, $[[[[c,a],b],c],[[c,a],a]]_{108}$,
$[[[[c,a],b],c],[[c,a],b]]_{109}$, $[[[[c,a],b],c],[[c,a],c]]_{110}$, $[[[[c,a],b],c],[[c,b],b]]_{111}$,
$[[[[c,a],b],c],[[c,b],c]]_{112}$, $[[[[c,a],c],c],[[b,a],a]]_{113}$, $[[[[c,a],c],c],[[b,a],b]]_{114}$,
$[[[[c,a],c],c],[[b,a],c]]_{115}$, $[[[[c,a],c],c],[[c,a],a]]_{116}$, $[[[[c,a],c],c],[[c,a],b]]_{117}$,
$[[[[c,a],c],c],[[c,a],c]]_{118}$, $[[[[c,a],c],c],[[c,b],b]]_{119}$, $[[[[c,a],c],c],[[c,b],c]]_{120}$,
$[[[[c,b],b],b],[[b,a],a]]_{121}$, $[[[[c,b],b],b],[[b,a],b]]_{122}$, $[[[[c,b],b],b],[[b,a],c]]_{123}$,
$[[[[c,b],b],b],[[c,a],a]]_{124}$, $[[[[c,b],b],b],[[c,a],b]]_{125}$, $[[[[c,b],b],b],[[c,a],c]]_{126}$,

$[[[[c,b],b],b],[[c,b],b]]_{127}$, $[[[[c,b],b],b],[[c,b],c]]_{128}$, $[[[[c,b],b],c],[[b,a],a]]_{129}$,
$[[[[c,b],b],c],[[b,a],b]]_{130}$, $[[[[c,b],b],c],[[b,a],c]]_{131}$, $[[[[c,b],b],c],[[c,a],a]]_{132}$,
$[[[[c,b],b],c],[[c,a],b]]_{133}$, $[[[[c,b],b],c],[[c,a],c]]_{134}$, $[[[[c,b],b],c],[[c,b],b]]_{135}$,
$[[[[c,b],b],c],[[c,b],c]]_{136}$, $[[[[c,b],c],c],[[b,a],a]]_{137}$, $[[[[c,b],c],c],[[b,a],b]]_{138}$,
$[[[[c,b],c],c],[[b,a],c]]_{139}$, $[[[[c,b],c],c],[[c,a],a]]_{140}$, $[[[[c,b],c],c],[[c,a],b]]_{141}$,
$[[[[c,b],c],c],[[c,a],c]]_{142}$, $[[[[c,b],c],c],[[c,b],b]]_{143}$, $[[[[c,b],c],c],[[c,b],c]]_{144}$,
$[[[[b,a],a],[b,a]],[b,a]]_{145}$, $[[[[b,a],a],[b,a]],[c,a]]_{146}$, $[[[[b,a],a],[b,a]],[c,b]]_{147}$,
$[[[[b,a],a],[c,a]],[c,a]]_{148}$, $[[[[b,a],a],[c,a]],[c,b]]_{149}$, $[[[[b,a],a],[c,b]],[c,b]]_{150}$,
$[[[[b,a],b],[b,a]],[b,a]]_{151}$, $[[[[b,a],b],[b,a]],[c,a]]_{152}$, $[[[[b,a],b],[b,a]],[c,b]]_{153}$,
$[[[[b,a],b],[c,a]],[c,a]]_{154}$, $[[[[b,a],b],[c,a]],[c,b]]_{155}$, $[[[[b,a],b],[c,b]],[c,b]]_{156}$,
$[[[[b,a],c],[b,a]],[b,a]]_{157}$, $[[[[b,a],c],[b,a]],[c,a]]_{158}$, $[[[[b,a],c],[b,a]],[c,b]]_{159}$,
$[[[[b,a],c],[c,a]],[c,a]]_{160}$, $[[[[b,a],c],[c,a]],[c,b]]_{161}$, $[[[[b,a],c],[c,b]],[c,b]]_{162}$,
$[[[[c,a],a],[b,a]],[b,a]]_{163}$, $[[[[c,a],a],[b,a]],[c,a]]_{164}$, $[[[[c,a],a],[b,a]],[c,b]]_{165}$,
$[[[[c,a],a],[c,a]],[c,a]]_{166}$, $[[[[c,a],a],[c,a]],[c,b]]_{167}$, $[[[[c,a],a],[c,b]],[c,b]]_{168}$,
$[[[[c,a],b],[b,a]],[b,a]]_{169}$, $[[[[c,a],b],[b,a]],[c,a]]_{170}$, $[[[[c,a],b],[b,a]],[c,b]]_{171}$,
$[[[[c,a],b],[c,a]],[c,a]]_{172}$, $[[[[c,a],b],[c,a]],[c,b]]_{173}$, $[[[[c,a],b],[c,b]],[c,b]]_{174}$,
$[[[[c,a],c],[b,a]],[b,a]]_{175}$, $[[[[c,a],c],[b,a]],[c,a]]_{176}$, $[[[[c,a],c],[b,a]],[c,b]]_{177}$,
$[[[[c,a],c],[c,a]],[c,a]]_{178}$, $[[[[c,a],c],[c,a]],[c,b]]_{179}$, $[[[[c,a],c],[c,b]],[c,b]]_{180}$,
$[[[[c,b],b],[b,a]],[b,a]]_{181}$, $[[[[c,b],b],[b,a]],[c,a]]_{182}$, $[[[[c,b],b],[b,a]],[c,b]]_{183}$,
$[[[[c,b],b],[c,a]],[c,a]]_{184}$, $[[[[c,b],b],[c,a]],[c,b]]_{185}$, $[[[[c,b],b],[c,b]],[c,b]]_{186}$,
$[[[[c,b],c],[b,a]],[b,a]]_{187}$, $[[[[c,b],c],[b,a]],[c,a]]_{188}$, $[[[[c,b],c],[b,a]],[c,b]]_{189}$,
$[[[[c,b],c],[c,a]],[c,a]]_{190}$, $[[[[c,b],c],[c,a]],[c,b]]_{191}$, $[[[[c,b],c],[c,b]],[c,b]]_{192}$,
$[[[[[b,a],a],a],a],[b,a]]_{193}$, $[[[[[b,a],a],a],a],[c,a]]_{194}$, $[[[[[b,a],a],a],a],[c,b]]_{195}$,
$[[[[[b,a],a],a],b],[b,a]]_{196}$, $[[[[[b,a],a],a],b],[c,a]]_{197}$, $[[[[[b,a],a],a],b],[c,b]]_{198}$,
$[[[[[b,a],a],a],c],[b,a]]_{199}$, $[[[[[b,a],a],a],c],[c,a]]_{200}$, $[[[[[b,a],a],a],c],[c,b]]_{201}$,
$[[[[[b,a],a],b],b],[b,a]]_{202}$, $[[[[[b,a],a],b],b],[c,a]]_{203}$, $[[[[[b,a],a],b],b],[c,b]]_{204}$,
$[[[[[b,a],a],b],c],[b,a]]_{205}$, $[[[[[b,a],a],b],c],[c,a]]_{206}$, $[[[[[b,a],a],b],c],[c,b]]_{207}$,
$[[[[[b,a],a],c],c],[b,a]]_{208}$, $[[[[[b,a],a],c],c],[c,a]]_{209}$, $[[[[[b,a],a],c],c],[c,b]]_{210}$,
$[[[[[b,a],b],b],b],[b,a]]_{211}$, $[[[[[b,a],b],b],b],[c,a]]_{212}$, $[[[[[b,a],b],b],b],[c,b]]_{213}$,
$[[[[[b,a],b],b],c],[b,a]]_{214}$, $[[[[[b,a],b],b],c],[c,a]]_{215}$, $[[[[[b,a],b],b],c],[c,b]]_{216}$,
$[[[[[b,a],b],c],c],[b,a]]_{217}$, $[[[[[b,a],b],c],c],[c,a]]_{218}$, $[[[[[b,a],b],c],c],[c,b]]_{219}$,
$[[[[[b,a],c],c],c],[b,a]]_{220}$, $[[[[[b,a],c],c],c],[c,a]]_{221}$, $[[[[[b,a],c],c],c],[c,b]]_{222}$,
$[[[[[c,a],a],a],a],[b,a]]_{223}$, $[[[[[c,a],a],a],a],[c,a]]_{224}$, $[[[[[c,a],a],a],a],[c,b]]_{225}$,
$[[[[[c,a],a],a],b],[b,a]]_{226}$, $[[[[[c,a],a],a],b],[c,a]]_{227}$, $[[[[[c,a],a],a],b],[c,b]]_{228}$,
$[[[[[c,a],a],a],c],[b,a]]_{229}$, $[[[[[c,a],a],a],c],[c,a]]_{230}$, $[[[[[c,a],a],a],c],[c,b]]_{231}$,
$[[[[[c,a],a],b],b],[b,a]]_{232}$, $[[[[[c,a],a],b],b],[c,a]]_{233}$, $[[[[[c,a],a],b],b],[c,b]]_{234}$,
$[[[[[c,a],a],b],c],[b,a]]_{235}$, $[[[[[c,a],a],b],c],[c,a]]_{236}$, $[[[[[c,a],a],b],c],[c,b]]_{237}$,

$$
\begin{array}{lll}
[[[[[c,a],a],c],c],[b,a]]_{238}, & [[[[[c,a],a],c],c],[c,a]]_{239}, & [[[[[c,a],a],c],c],[c,b]]_{240},\\
[[[[[c,a],b],b],b],[b,a]]_{241}, & [[[[[c,a],b],b],b],[c,a]]_{242}, & [[[[[c,a],b],b],b],[c,b]]_{243},\\
[[[[[c,a],b],b],c],[b,a]]_{244}, & [[[[[c,a],b],b],c],[c,a]]_{245}, & [[[[[c,a],b],b],c],[c,b]]_{246},\\
[[[[[c,a],b],c],c],[b,a]]_{247}, & [[[[[c,a],b],c],c],[c,a]]_{248}, & [[[[[c,a],b],c],c],[c,b]]_{249},\\
[[[[[c,a],c],c],c],[b,a]]_{250}, & [[[[[c,a],c],c],c],[c,a]]_{251}, & [[[[[c,a],c],c],c],[c,b]]_{252},\\
[[[[[c,b],b],b],b],[b,a]]_{253}, & [[[[[c,b],b],b],b],[c,a]]_{254}, & [[[[[c,b],b],b],b],[c,b]]_{255},\\
[[[[[c,b],b],b],c],[b,a]]_{256}, & [[[[[c,b],b],b],c],[c,a]]_{257}, & [[[[[c,b],b],b],c],[c,b]]_{258},\\
[[[[[c,b],b],c],c],[b,a]]_{259}, & [[[[[c,b],b],c],c],[c,a]]_{260}, & [[[[[c,b],b],c],c],[c,b]]_{261},\\
[[[[[c,b],c],c],c],[b,a]]_{262}, & [[[[[c,b],c],c],c],[c,a]]_{263}, & [[[[[c,b],c],c],c],[c,b]]_{264},\\
[[[[[[b,a],a],a],a],a],a]_{265}, & [[[[[[b,a],a],a],a],a],b]_{266}, & [[[[[[b,a],a],a],a],a],c]_{267},\\
[[[[[[b,a],a],a],a],b],b]_{268}, & [[[[[[b,a],a],a],a],b],c]_{269}, & [[[[[[b,a],a],a],a],c],c]_{270},\\
[[[[[[b,a],a],a],b],b],b]_{271}, & [[[[[[b,a],a],a],b],b],c]_{272}, & [[[[[[b,a],a],a],b],c],c]_{273},\\
[[[[[[b,a],a],a],c],c],c]_{274}, & [[[[[[b,a],a],b],b],b],b]_{275}, & [[[[[[b,a],a],b],b],b],c]_{276},\\
[[[[[[b,a],a],b],b],c],c]_{277}, & [[[[[[b,a],a],b],c],c],c]_{278}, & [[[[[[b,a],a],c],c],c],c]_{279},\\
[[[[[[b,a],b],b],b],b],b]_{280}, & [[[[[[b,a],b],b],b],b],c]_{281}, & [[[[[[b,a],b],b],b],c],c]_{282},\\
[[[[[[b,a],b],b],c],c],c]_{283}, & [[[[[[b,a],b],c],c],c],c]_{284}, & [[[[[[b,a],c],c],c],c],c]_{285},\\
[[[[[[c,a],a],a],a],a],a]_{286}, & [[[[[[c,a],a],a],a],a],b]_{287}, & [[[[[[c,a],a],a],a],a],c]_{288},\\
[[[[[[c,a],a],a],a],b],b]_{289}, & [[[[[[c,a],a],a],a],b],c]_{290}, & [[[[[[c,a],a],a],a],c],c]_{291},\\
[[[[[[c,a],a],a],b],b],b]_{292}, & [[[[[[c,a],a],a],b],b],c]_{293}, & [[[[[[c,a],a],a],b],c],c]_{294},\\
[[[[[[c,a],a],a],c],c],c]_{295}, & [[[[[[c,a],a],b],b],b],b]_{296}, & [[[[[[c,a],a],b],b],b],c]_{297},\\
[[[[[[c,a],a],b],b],c],c]_{298}, & [[[[[[c,a],a],b],c],c],c]_{299}, & [[[[[[c,a],a],c],c],c],c]_{300},\\
[[[[[[c,a],b],b],b],b],b]_{301}, & [[[[[[c,a],b],b],b],b],c]_{302}, & [[[[[[c,a],b],b],b],c],c]_{303},\\
[[[[[[c,a],b],b],c],c],c]_{304}, & [[[[[[c,a],b],c],c],c],c]_{305}, & [[[[[[c,a],c],c],c],c],c]_{306},\\
[[[[[[c,b],b],b],b],b],b]_{307}, & [[[[[[c,b],b],b],b],b],c]_{308}, & [[[[[[c,b],b],b],b],c],c]_{309},\\
[[[[[[c,b],b],b],c],c],c]_{310}, & [[[[[[c,b],b],c],c],c],c]_{311}, & [[[[[[c,b],c],c],c],c],c]_{312}.
\end{array}
$$

D.4 Degree 8

$$
\begin{array}{ll}
[[[c,b],[b,a]],[[c,a],[b,a]]]_{1}, & [[[c,b],[c,a]],[[c,a],[b,a]]]_{2},\\
[[[c,b],[c,a]],[[c,b],[b,a]]]_{3}, & [[[[b,a],a],a],[[c,a],[b,a]]]_{4},\\
[[[[b,a],a],a],[[c,b],[b,a]]]_{5}, & [[[[b,a],a],a],[[c,b],[c,a]]]_{6},\\
[[[[b,a],a],b],[[c,a],[b,a]]]_{7}, & [[[[b,a],a],b],[[c,b],[b,a]]]_{8},\\
[[[[b,a],a],b],[[c,b],[c,a]]]_{9}, & [[[[b,a],a],b],[[[b,a],a],a]]_{10},\\
[[[[b,a],a],c],[[c,a],[b,a]]]_{11}, & [[[[b,a],a],c],[[c,b],[b,a]]]_{12},\\
[[[[b,a],a],c],[[c,b],[c,a]]]_{13}, & [[[[b,a],a],c],[[[b,a],a],a]]_{14},\\
[[[[b,a],a],c],[[[b,a],a],b]]_{15}, & [[[[b,a],b],b],[[c,a],[b,a]]]_{16},\\
[[[[b,a],b],b],[[c,b],[b,a]]]_{17}, & [[[[b,a],b],b],[[c,b],[c,a]]]_{18},
\end{array}
$$

$[[[[b,a],b],b],[[[b,a],a],a]]_{19}$, $[[[[b,a],b],b],[[[b,a],a],b]]_{20}$,
$[[[[b,a],b],b],[[[b,a],a],c]]_{21}$, $[[[[b,a],b],c],[[c,a],[b,a]]]_{22}$,
$[[[[b,a],b],c],[[c,b],[b,a]]]_{23}$, $[[[[b,a],b],c],[[c,b],[c,a]]]_{24}$,
$[[[[b,a],b],c],[[[b,a],a],a]]_{25}$, $[[[[b,a],b],c],[[[b,a],a],b]]_{26}$,
$[[[[b,a],b],c],[[[b,a],a],c]]_{27}$, $[[[[b,a],b],c],[[[b,a],b],b]]_{28}$,
$[[[[b,a],c],c],[[c,a],[b,a]]]_{29}$, $[[[[b,a],c],c],[[c,b],[b,a]]]_{30}$,
$[[[[b,a],c],c],[[c,b],[c,a]]]_{31}$, $[[[[b,a],c],c],[[[b,a],a],a]]_{32}$,
$[[[[b,a],c],c],[[[b,a],a],b]]_{33}$, $[[[[b,a],c],c],[[[b,a],a],c]]_{34}$,
$[[[[b,a],c],c],[[[b,a],b],b]]_{35}$, $[[[[b,a],c],c],[[[b,a],b],c]]_{36}$,
$[[[[c,a],a],a],[[c,a],[b,a]]]_{37}$, $[[[[c,a],a],a],[[c,b],[b,a]]]_{38}$,
$[[[[c,a],a],a],[[c,b],[c,a]]]_{39}$, $[[[[c,a],a],a],[[[b,a],a],a]]_{40}$,
$[[[[c,a],a],a],[[[b,a],a],b]]_{41}$, $[[[[c,a],a],a],[[[b,a],a],c]]_{42}$,
$[[[[c,a],a],a],[[[b,a],b],b]]_{43}$, $[[[[c,a],a],a],[[[b,a],b],c]]_{44}$,
$[[[[c,a],a],a],[[[b,a],c],c]]_{45}$, $[[[[c,a],a],b],[[c,a],[b,a]]]_{46}$,
$[[[[c,a],a],b],[[c,b],[b,a]]]_{47}$, $[[[[c,a],a],b],[[c,b],[c,a]]]_{48}$,
$[[[[c,a],a],b],[[[b,a],a],a]]_{49}$, $[[[[c,a],a],b],[[[b,a],a],b]]_{50}$,
$[[[[c,a],a],b],[[[b,a],a],c]]_{51}$, $[[[[c,a],a],b],[[[b,a],b],b]]_{52}$,
$[[[[c,a],a],b],[[[b,a],b],c]]_{53}$, $[[[[c,a],a],b],[[[b,a],c],c]]_{54}$,
$[[[[c,a],a],b],[[[c,a],a],a]]_{55}$, $[[[[c,a],a],c],[[c,a],[b,a]]]_{56}$,
$[[[[c,a],a],c],[[c,b],[b,a]]]_{57}$, $[[[[c,a],a],c],[[c,b],[c,a]]]_{58}$,
$[[[[c,a],a],c],[[[b,a],a],a]]_{59}$, $[[[[c,a],a],c],[[[b,a],a],b]]_{60}$,
$[[[[c,a],a],c],[[[b,a],a],c]]_{61}$, $[[[[c,a],a],c],[[[b,a],b],b]]_{62}$,
$[[[[c,a],a],c],[[[b,a],b],c]]_{63}$, $[[[[c,a],a],c],[[[b,a],c],c]]_{64}$,
$[[[[c,a],a],c],[[[c,a],a],a]]_{65}$, $[[[[c,a],a],c],[[[c,a],a],b]]_{66}$,
$[[[[c,a],b],b],[[c,a],[b,a]]]_{67}$, $[[[[c,a],b],b],[[c,b],[b,a]]]_{68}$,
$[[[[c,a],b],b],[[c,b],[c,a]]]_{69}$, $[[[[c,a],b],b],[[[b,a],a],a]]_{70}$,
$[[[[c,a],b],b],[[[b,a],a],b]]_{71}$, $[[[[c,a],b],b],[[[b,a],a],c]]_{72}$,
$[[[[c,a],b],b],[[[b,a],b],b]]_{73}$, $[[[[c,a],b],b],[[[b,a],b],c]]_{74}$,
$[[[[c,a],b],b],[[[b,a],c],c]]_{75}$, $[[[[c,a],b],b],[[[c,a],a],a]]_{76}$,
$[[[[c,a],b],b],[[[c,a],a],b]]_{77}$, $[[[[c,a],b],b],[[[c,a],a],c]]_{78}$,
$[[[[c,a],b],c],[[c,a],[b,a]]]_{79}$, $[[[[c,a],b],c],[[c,b],[b,a]]]_{80}$,
$[[[[c,a],b],c],[[c,b],[c,a]]]_{81}$, $[[[[c,a],b],c],[[[b,a],a],a]]_{82}$,
$[[[[c,a],b],c],[[[b,a],a],b]]_{83}$, $[[[[c,a],b],c],[[[b,a],a],c]]_{84}$,
$[[[[c,a],b],c],[[[b,a],b],b]]_{85}$, $[[[[c,a],b],c],[[[b,a],b],c]]_{86}$,
$[[[[c,a],b],c],[[[b,a],c],c]]_{87}$, $[[[[c,a],b],c],[[[c,a],a],a]]_{88}$,
$[[[[c,a],b],c],[[[c,a],a],b]]_{89}$, $[[[[c,a],b],c],[[[c,a],a],c]]_{90}$,
$[[[[c,a],b],c],[[[c,a],b],b]]_{91}$, $[[[[c,a],c],c],[[c,a],[b,a]]]_{92}$,

$[[[[c,a],c],c],[[c,b],[b,a]]]_{93}$, $[[[[c,a],c],c],[[c,b],[c,a]]]_{94}$,
$[[[[c,a],c],c],[[[b,a],a],a]]_{95}$, $[[[[c,a],c],c],[[[b,a],a],b]]_{96}$,
$[[[[c,a],c],c],[[[b,a],a],c]]_{97}$, $[[[[c,a],c],c],[[[b,a],b],b]]_{98}$,
$[[[[c,a],c],c],[[[b,a],b],c]]_{99}$, $[[[[c,a],c],c],[[[b,a],c],c]]_{100}$,
$[[[[c,a],c],c],[[[c,a],a],a]]_{101}$, $[[[[c,a],c],c],[[[c,a],a],b]]_{102}$,
$[[[[c,a],c],c],[[[c,a],a],c]]_{103}$, $[[[[c,a],c],c],[[[c,a],b],b]]_{104}$,
$[[[[c,a],c],c],[[[c,a],b],c]]_{105}$, $[[[[c,b],b],b],[[c,a],[b,a]]]_{106}$,
$[[[[c,b],b],b],[[c,b],[b,a]]]_{107}$, $[[[[c,b],b],b],[[c,b],[c,a]]]_{108}$,
$[[[[c,b],b],b],[[[b,a],a],a]]_{109}$, $[[[[c,b],b],b],[[[b,a],a],b]]_{110}$,
$[[[[c,b],b],b],[[[b,a],a],c]]_{111}$, $[[[[c,b],b],b],[[[b,a],b],b]]_{112}$,
$[[[[c,b],b],b],[[[b,a],b],c]]_{113}$, $[[[[c,b],b],b],[[[b,a],c],c]]_{114}$,
$[[[[c,b],b],b],[[[c,a],a],a]]_{115}$, $[[[[c,b],b],b],[[[c,a],a],b]]_{116}$,
$[[[[c,b],b],b],[[[c,a],a],c]]_{117}$, $[[[[c,b],b],b],[[[c,a],b],b]]_{118}$,
$[[[[c,b],b],b],[[[c,a],b],c]]_{119}$, $[[[[c,b],b],b],[[[c,a],c],c]]_{120}$,
$[[[[c,b],b],c],[[c,a],[b,a]]]_{121}$, $[[[[c,b],b],c],[[c,b],[b,a]]]_{122}$,
$[[[[c,b],b],c],[[c,b],[c,a]]]_{123}$, $[[[[c,b],b],c],[[[b,a],a],a]]_{124}$,
$[[[[c,b],b],c],[[[b,a],a],b]]_{125}$, $[[[[c,b],b],c],[[[b,a],a],c]]_{126}$,
$[[[[c,b],b],c],[[[b,a],b],b]]_{127}$, $[[[[c,b],b],c],[[[b,a],b],c]]_{128}$,
$[[[[c,b],b],c],[[[b,a],c],c]]_{129}$, $[[[[c,b],b],c],[[[c,a],a],a]]_{130}$,
$[[[[c,b],b],c],[[[c,a],a],b]]_{131}$, $[[[[c,b],b],c],[[[c,a],a],c]]_{132}$,
$[[[[c,b],b],c],[[[c,a],b],b]]_{133}$, $[[[[c,b],b],c],[[[c,a],b],c]]_{134}$,
$[[[[c,b],b],c],[[[c,a],c],c]]_{135}$, $[[[[c,b],b],c],[[[c,b],b],b]]_{136}$,
$[[[[c,b],c],c],[[c,a],[b,a]]]_{137}$, $[[[[c,b],c],c],[[c,b],[b,a]]]_{138}$,
$[[[[c,b],c],c],[[c,b],[c,a]]]_{139}$, $[[[[c,b],c],c],[[[b,a],a],a]]_{140}$,
$[[[[c,b],c],c],[[[b,a],a],b]]_{141}$, $[[[[c,b],c],c],[[[b,a],a],c]]_{142}$,
$[[[[c,b],c],c],[[[b,a],b],b]]_{143}$, $[[[[c,b],c],c],[[[b,a],b],c]]_{144}$,
$[[[[c,b],c],c],[[[b,a],c],c]]_{145}$, $[[[[c,b],c],c],[[[c,a],a],a]]_{146}$,
$[[[[c,b],c],c],[[[c,a],a],b]]_{147}$, $[[[[c,b],c],c],[[[c,a],a],c]]_{148}$,
$[[[[c,b],c],c],[[[c,a],b],b]]_{149}$, $[[[[c,b],c],c],[[[c,a],b],c]]_{150}$,
$[[[[c,b],c],c],[[[c,a],c],c]]_{151}$, $[[[[c,b],c],c],[[[c,b],b],b]]_{152}$,
$[[[[c,b],c],c],[[[c,b],b],c]]_{153}$, $[[[[b,a],a],[b,a]],[[b,a],a]]_{154}$,
$[[[[b,a],a],[b,a]],[[b,a],b]]_{155}$, $[[[[b,a],a],[b,a]],[[b,a],c]]_{156}$,
$[[[[b,a],a],[b,a]],[[c,a],a]]_{157}$, $[[[[b,a],a],[b,a]],[[c,a],b]]_{158}$,
$[[[[b,a],a],[b,a]],[[c,a],c]]_{159}$, $[[[[b,a],a],[b,a]],[[c,b],b]]_{160}$,
$[[[[b,a],a],[b,a]],[[c,b],c]]_{161}$, $[[[[b,a],a],[c,a]],[[b,a],a]]_{162}$,
$[[[[b,a],a],[c,a]],[[b,a],b]]_{163}$, $[[[[b,a],a],[c,a]],[[b,a],c]]_{164}$,
$[[[[b,a],a],[c,a]],[[c,a],a]]_{165}$, $[[[[b,a],a],[c,a]],[[c,a],b]]_{166}$,

$[[[[b,a],a],[c,a]],[[c,a],c]]_{167}$, $[[[[b,a],a],[c,a]],[[c,b],b]]_{168}$,
$[[[[b,a],a],[c,a]],[[c,b],c]]_{169}$, $[[[[b,a],a],[c,b]],[[b,a],a]]_{170}$,
$[[[[b,a],a],[c,b]],[[b,a],b]]_{171}$, $[[[[b,a],a],[c,b]],[[b,a],c]]_{172}$,
$[[[[b,a],a],[c,b]],[[c,a],a]]_{173}$, $[[[[b,a],a],[c,b]],[[c,a],b]]_{174}$,
$[[[[b,a],a],[c,b]],[[c,a],c]]_{175}$, $[[[[b,a],a],[c,b]],[[c,b],b]]_{176}$,
$[[[[b,a],a],[c,b]],[[c,b],c]]_{177}$, $[[[[b,a],b],[b,a]],[[b,a],a]]_{178}$,
$[[[[b,a],b],[b,a]],[[b,a],b]]_{179}$, $[[[[b,a],b],[b,a]],[[b,a],c]]_{180}$,
$[[[[b,a],b],[b,a]],[[c,a],a]]_{181}$, $[[[[b,a],b],[b,a]],[[c,a],b]]_{182}$,
$[[[[b,a],b],[b,a]],[[c,a],c]]_{183}$, $[[[[b,a],b],[b,a]],[[c,b],b]]_{184}$,
$[[[[b,a],b],[b,a]],[[c,b],c]]_{185}$, $[[[[b,a],b],[c,a]],[[b,a],a]]_{186}$,
$[[[[b,a],b],[c,a]],[[b,a],b]]_{187}$, $[[[[b,a],b],[c,a]],[[b,a],c]]_{188}$,
$[[[[b,a],b],[c,a]],[[c,a],a]]_{189}$, $[[[[b,a],b],[c,a]],[[c,a],b]]_{190}$,
$[[[[b,a],b],[c,a]],[[c,a],c]]_{191}$, $[[[[b,a],b],[c,a]],[[c,b],b]]_{192}$,
$[[[[b,a],b],[c,a]],[[c,b],c]]_{193}$, $[[[[b,a],b],[c,b]],[[b,a],a]]_{194}$,
$[[[[b,a],b],[c,b]],[[b,a],b]]_{195}$, $[[[[b,a],b],[c,b]],[[b,a],c]]_{196}$,
$[[[[b,a],b],[c,b]],[[c,a],a]]_{197}$, $[[[[b,a],b],[c,b]],[[c,a],b]]_{198}$,
$[[[[b,a],b],[c,b]],[[c,a],c]]_{199}$, $[[[[b,a],b],[c,b]],[[c,b],b]]_{200}$,
$[[[[b,a],b],[c,b]],[[c,b],c]]_{201}$, $[[[[b,a],c],[b,a]],[[b,a],a]]_{202}$,
$[[[[b,a],c],[b,a]],[[b,a],b]]_{203}$, $[[[[b,a],c],[b,a]],[[b,a],c]]_{204}$,
$[[[[b,a],c],[b,a]],[[c,a],a]]_{205}$, $[[[[b,a],c],[b,a]],[[c,a],b]]_{206}$,
$[[[[b,a],c],[b,a]],[[c,a],c]]_{207}$, $[[[[b,a],c],[b,a]],[[c,b],b]]_{208}$,
$[[[[b,a],c],[b,a]],[[c,b],c]]_{209}$, $[[[[b,a],c],[c,a]],[[b,a],a]]_{210}$,
$[[[[b,a],c],[c,a]],[[b,a],b]]_{211}$, $[[[[b,a],c],[c,a]],[[b,a],c]]_{212}$,
$[[[[b,a],c],[c,a]],[[c,a],a]]_{213}$, $[[[[b,a],c],[c,a]],[[c,a],b]]_{214}$,
$[[[[b,a],c],[c,a]],[[c,a],c]]_{215}$, $[[[[b,a],c],[c,a]],[[c,b],b]]_{216}$,
$[[[[b,a],c],[c,a]],[[c,b],c]]_{217}$, $[[[[b,a],c],[c,b]],[[b,a],a]]_{218}$,
$[[[[b,a],c],[c,b]],[[b,a],b]]_{219}$, $[[[[b,a],c],[c,b]],[[b,a],c]]_{220}$,
$[[[[b,a],c],[c,b]],[[c,a],a]]_{221}$, $[[[[b,a],c],[c,b]],[[c,a],b]]_{222}$,
$[[[[b,a],c],[c,b]],[[c,a],c]]_{223}$, $[[[[b,a],c],[c,b]],[[c,b],b]]_{224}$,
$[[[[b,a],c],[c,b]],[[c,b],c]]_{225}$, $[[[[c,a],a],[b,a]],[[b,a],a]]_{226}$,
$[[[[c,a],a],[b,a]],[[b,a],b]]_{227}$, $[[[[c,a],a],[b,a]],[[b,a],c]]_{228}$,
$[[[[c,a],a],[b,a]],[[c,a],a]]_{229}$, $[[[[c,a],a],[b,a]],[[c,a],b]]_{230}$,
$[[[[c,a],a],[b,a]],[[c,a],c]]_{231}$, $[[[[c,a],a],[b,a]],[[c,b],b]]_{232}$,
$[[[[c,a],a],[b,a]],[[c,b],c]]_{233}$, $[[[[c,a],a],[c,a]],[[b,a],a]]_{234}$,
$[[[[c,a],a],[c,a]],[[b,a],b]]_{235}$, $[[[[c,a],a],[c,a]],[[b,a],c]]_{236}$,
$[[[[c,a],a],[c,a]],[[c,a],a]]_{237}$, $[[[[c,a],a],[c,a]],[[c,a],b]]_{238}$,
$[[[[c,a],a],[c,a]],[[c,a],c]]_{239}$, $[[[[c,a],a],[c,a]],[[c,b],b]]_{240}$,

$$[[[[c,a],a],[c,a]],[[c,b],c]]_{241}, \qquad [[[[c,a],a],[c,b]],[[b,a],a]]_{242},$$
$$[[[[c,a],a],[c,b]],[[b,a],b]]_{243}, \qquad [[[[c,a],a],[c,b]],[[b,a],c]]_{244},$$
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$[[[[[b,a],b],b],[b,a]],[b,a]]_{571}$, $[[[[[b,a],b],b],[b,a]],[c,a]]_{572}$,
$[[[[[b,a],b],b],[b,a]],[c,b]]_{573}$, $[[[[[b,a],b],b],[c,a]],[c,a]]_{574}$,
$[[[[[b,a],b],b],[c,a]],[c,b]]_{575}$, $[[[[[b,a],b],b],[c,b]],[c,b]]_{576}$,
$[[[[[b,a],b],c],[b,a]],[b,a]]_{577}$, $[[[[[b,a],b],c],[b,a]],[c,a]]_{578}$,
$[[[[[b,a],b],c],[b,a]],[c,b]]_{579}$, $[[[[[b,a],b],c],[c,a]],[c,a]]_{580}$,
$[[[[[b,a],b],c],[c,a]],[c,b]]_{581}$, $[[[[[b,a],b],c],[c,b]],[c,b]]_{582}$,
$[[[[[b,a],c],c],[b,a]],[b,a]]_{583}$, $[[[[[b,a],c],c],[b,a]],[c,a]]_{584}$,
$[[[[[b,a],c],c],[b,a]],[c,b]]_{585}$, $[[[[[b,a],c],c],[c,a]],[c,a]]_{586}$,
$[[[[[b,a],c],c],[c,a]],[c,b]]_{587}$, $[[[[[b,a],c],c],[c,b]],[c,b]]_{588}$,
$[[[[[c,a],a],a],[b,a]],[b,a]]_{589}$, $[[[[[c,a],a],a],[b,a]],[c,a]]_{590}$,
$[[[[[c,a],a],a],[b,a]],[c,b]]_{591}$, $[[[[[c,a],a],a],[c,a]],[c,a]]_{592}$,
$[[[[[c,a],a],a],[c,a]],[c,b]]_{593}$, $[[[[[c,a],a],a],[c,b]],[c,b]]_{594}$,
$[[[[[c,a],a],b],[b,a]],[b,a]]_{595}$, $[[[[[c,a],a],b],[b,a]],[c,a]]_{596}$,
$[[[[[c,a],a],b],[b,a]],[c,b]]_{597}$, $[[[[[c,a],a],b],[c,a]],[c,a]]_{598}$,
$[[[[[c,a],a],b],[c,a]],[c,b]]_{599}$, $[[[[[c,a],a],b],[c,b]],[c,b]]_{600}$,
$[[[[[c,a],a],c],[b,a]],[b,a]]_{601}$, $[[[[[c,a],a],c],[b,a]],[c,a]]_{602}$,
$[[[[[c,a],a],c],[b,a]],[c,b]]_{603}$, $[[[[[c,a],a],c],[c,a]],[c,a]]_{604}$,
$[[[[[c,a],a],c],[c,a]],[c,b]]_{605}$, $[[[[[c,a],a],c],[c,b]],[c,b]]_{606}$,
$[[[[[c,a],b],b],[b,a]],[b,a]]_{607}$, $[[[[[c,a],b],b],[b,a]],[c,a]]_{608}$,
$[[[[[c,a],b],b],[b,a]],[c,b]]_{609}$, $[[[[[c,a],b],b],[c,a]],[c,a]]_{610}$,

$[[[[[c,a],b],b],[c,a]],[c,b]]_{611}$, $[[[[[c,a],b],b],[c,b]],[c,b]]_{612}$,
$[[[[[c,a],b],c],[b,a]],[b,a]]_{613}$, $[[[[[c,a],b],c],[b,a]],[c,a]]_{614}$,
$[[[[[c,a],b],c],[b,a]],[c,b]]_{615}$, $[[[[[c,a],b],c],[c,a]],[c,a]]_{616}$,
$[[[[[c,a],b],c],[c,a]],[c,b]]_{617}$, $[[[[[c,a],b],c],[c,b]],[c,b]]_{618}$,
$[[[[[c,a],c],c],[b,a]],[b,a]]_{619}$, $[[[[[c,a],c],c],[b,a]],[c,a]]_{620}$,
$[[[[[c,a],c],c],[b,a]],[c,b]]_{621}$, $[[[[[c,a],c],c],[c,a]],[c,a]]_{622}$,
$[[[[[c,a],c],c],[c,a]],[c,b]]_{623}$, $[[[[[c,a],c],c],[c,b]],[c,b]]_{624}$,
$[[[[[c,b],b],b],[b,a]],[b,a]]_{625}$, $[[[[[c,b],b],b],[b,a]],[c,a]]_{626}$,
$[[[[[c,b],b],b],[b,a]],[c,b]]_{627}$, $[[[[[c,b],b],b],[c,a]],[c,a]]_{628}$,
$[[[[[c,b],b],b],[c,a]],[c,b]]_{629}$, $[[[[[c,b],b],b],[c,b]],[c,b]]_{630}$,
$[[[[[c,b],b],c],[b,a]],[b,a]]_{631}$, $[[[[[c,b],b],c],[b,a]],[c,a]]_{632}$,
$[[[[[c,b],b],c],[b,a]],[c,b]]_{633}$, $[[[[[c,b],b],c],[c,a]],[c,a]]_{634}$,
$[[[[[c,b],b],c],[c,a]],[c,b]]_{635}$, $[[[[[c,b],b],c],[c,b]],[c,b]]_{636}$,
$[[[[[c,b],c],c],[b,a]],[b,a]]_{637}$, $[[[[[c,b],c],c],[b,a]],[c,a]]_{638}$,
$[[[[[c,b],c],c],[b,a]],[c,b]]_{639}$, $[[[[[c,b],c],c],[c,a]],[c,a]]_{640}$,
$[[[[[c,b],c],c],[c,a]],[c,b]]_{641}$, $[[[[[c,b],c],c],[c,b]],[c,b]]_{642}$,
$[[[[[[b,a],a],a],a],a],[b,a]]_{643}$, $[[[[[[b,a],a],a],a],a],[c,a]]_{644}$,
$[[[[[[b,a],a],a],a],a],[c,b]]_{645}$, $[[[[[[b,a],a],a],a],b],[b,a]]_{646}$,
$[[[[[[b,a],a],a],a],b],[c,a]]_{647}$, $[[[[[[b,a],a],a],a],b],[c,b]]_{648}$,
$[[[[[[b,a],a],a],a],c],[b,a]]_{649}$, $[[[[[[b,a],a],a],a],c],[c,a]]_{650}$,
$[[[[[[b,a],a],a],a],c],[c,b]]_{651}$, $[[[[[[b,a],a],a],b],b],[b,a]]_{652}$,
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$[[[[[[b,a],a],b],b],b],[c,b]]_{663}$, $[[[[[[b,a],a],b],b],c],[b,a]]_{664}$,
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$[[[[[[b,a],b],b],b],c],[c,a]]_{677}$, $[[[[[[b,a],b],b],b],c],[c,b]]_{678}$,
$[[[[[[b,a],b],b],c],c],[b,a]]_{679}$, $[[[[[[b,a],b],b],c],c],[c,a]]_{680}$,
$[[[[[[b,a],b],b],c],c],[c,b]]_{681}$, $[[[[[[b,a],b],c],c],c],[b,a]]_{682}$,
$[[[[[[b,a],b],c],c],c],[c,a]]_{683}$, $[[[[[[b,a],b],c],c],c],[c,b]]_{684}$,

$[[[[[[b,a],c],c],c],c],[b,a]]_{685}$, $[[[[[[b,a],c],c],c],c],[c,a]]_{686}$,

$[[[[[[b,a],c],c],c],c],[c,b]]_{687}$, $[[[[[[c,a],a],a],a],a],[b,a]]_{688}$,

$[[[[[[c,a],a],a],a],a],[c,a]]_{689}$, $[[[[[[c,a],a],a],a],a],[c,b]]_{690}$,

$[[[[[[c,a],a],a],a],b],[b,a]]_{691}$, $[[[[[[c,a],a],a],a],b],[c,a]]_{692}$,

$[[[[[[c,a],a],a],a],b],[c,b]]_{693}$, $[[[[[[c,a],a],a],a],c],[b,a]]_{694}$,

$[[[[[[c,a],a],a],a],c],[c,a]]_{695}$, $[[[[[[c,a],a],a],a],c],[c,b]]_{696}$,

$[[[[[[c,a],a],a],b],b],[b,a]]_{697}$, $[[[[[[c,a],a],a],b],b],[c,a]]_{698}$,

$[[[[[[c,a],a],a],b],b],[c,b]]_{699}$, $[[[[[[c,a],a],a],b],c],[b,a]]_{700}$,

$[[[[[[c,a],a],a],b],c],[c,a]]_{701}$, $[[[[[[c,a],a],a],b],c],[c,b]]_{702}$,

$[[[[[[c,a],a],a],c],c],[b,a]]_{703}$, $[[[[[[c,a],a],a],c],c],[c,a]]_{704}$,

$[[[[[[c,a],a],a],c],c],[c,b]]_{705}$, $[[[[[[c,a],a],b],b],b],[b,a]]_{706}$,

$[[[[[[c,a],a],b],b],b],[c,a]]_{707}$, $[[[[[[c,a],a],b],b],b],[c,b]]_{708}$,

$[[[[[[c,a],a],b],b],c],[b,a]]_{709}$, $[[[[[[c,a],a],b],b],c],[c,a]]_{710}$,

$[[[[[[c,a],a],b],b],c],[c,b]]_{711}$, $[[[[[[c,a],a],b],c],c],[b,a]]_{712}$,

$[[[[[[c,a],a],b],c],c],[c,a]]_{713}$, $[[[[[[c,a],a],b],c],c],[c,b]]_{714}$,

$[[[[[[c,a],a],c],c],c],[b,a]]_{715}$, $[[[[[[c,a],a],c],c],c],[c,a]]_{716}$,

$[[[[[[c,a],a],c],c],c],[c,b]]_{717}$, $[[[[[[c,a],b],b],b],b],[b,a]]_{718}$,

$[[[[[[c,a],b],b],b],b],[c,a]]_{719}$, $[[[[[[c,a],b],b],b],b],[c,b]]_{720}$,

$[[[[[[c,a],b],b],b],c],[b,a]]_{721}$, $[[[[[[c,a],b],b],b],c],[c,a]]_{722}$,

$[[[[[[c,a],b],b],b],c],[c,b]]_{723}$, $[[[[[[c,a],b],b],c],c],[b,a]]_{724}$,

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$[[[[[[c,a],b],c],c],c],[b,a]]_{727}$, $[[[[[[c,a],b],c],c],c],[c,a]]_{728}$,

$[[[[[[c,a],b],c],c],c],[c,b]]_{729}$, $[[[[[[c,a],c],c],c],c],[b,a]]_{730}$,

$[[[[[[c,a],c],c],c],c],[c,a]]_{731}$, $[[[[[[c,a],c],c],c],c],[c,b]]_{732}$,

$[[[[[[c,b],b],b],b],b],[b,a]]_{733}$, $[[[[[[c,b],b],b],b],b],[c,a]]_{734}$,

$[[[[[[c,b],b],b],b],b],[c,b]]_{735}$, $[[[[[[c,b],b],b],b],c],[b,a]]_{736}$,

$[[[[[[c,b],b],b],b],c],[c,a]]_{737}$, $[[[[[[c,b],b],b],b],c],[c,b]]_{738}$,

$[[[[[[c,b],b],b],c],c],[b,a]]_{739}$, $[[[[[[c,b],b],b],c],c],[c,a]]_{740}$,

$[[[[[[c,b],b],b],c],c],[c,b]]_{741}$, $[[[[[[c,b],b],c],c],c],[b,a]]_{742}$,

$[[[[[[c,b],b],c],c],c],[c,a]]_{743}$, $[[[[[[c,b],b],c],c],c],[c,b]]_{744}$,

$[[[[[[c,b],c],c],c],c],[b,a]]_{745}$, $[[[[[[c,b],c],c],c],c],[c,a]]_{746}$,

$[[[[[[c,b],c],c],c],c],[c,b]]_{747}$, $[[[[[[[b,a],a],a],a],a],a],a]_{748}$,

$[[[[[[[b,a],a],a],a],a],a],b]_{749}$, $[[[[[[[b,a],a],a],a],a],a],c]_{750}$,

$[[[[[[[b,a],a],a],a],a],b],b]_{751}$, $[[[[[[[b,a],a],a],a],a],b],c]_{752}$,

$[[[[[[[b,a],a],a],a],a],c],c]_{753}$, $[[[[[[[b,a],a],a],a],b],b],b]_{754}$,

$[[[[[[[b,a],a],a],a],b],b],c]_{755}$, $[[[[[[[b,a],a],a],a],b],c],c]_{756}$,

$[[[[[[[b,a],a],a],a],c],c],c]_{757}$, $[[[[[[[b,a],a],a],b],b],b],b]_{758}$,

$$[[[[[[[b,a],a],a],b],b],b],c]_{759}, \qquad [[[[[[[b,a],a],a],b],b],c],c]_{760},$$
$$[[[[[[[b,a],a],a],b],c],c],c]_{761}, \qquad [[[[[[[b,a],a],a],c],c],c],c]_{762},$$
$$[[[[[[[b,a],a],b],b],b],b],b]_{763}, \qquad [[[[[[[b,a],a],b],b],b],b],c]_{764},$$
$$[[[[[[[b,a],a],b],b],b],c],c]_{765}, \qquad [[[[[[[b,a],a],b],b],c],c],c]_{766},$$
$$[[[[[[[b,a],a],b],c],c],c],c]_{767}, \qquad [[[[[[[b,a],a],c],c],c],c],c]_{768},$$
$$[[[[[[[b,a],b],b],b],b],b],b]_{769}, \qquad [[[[[[[b,a],b],b],b],b],b],c]_{770},$$
$$[[[[[[[b,a],b],b],b],b],c],c]_{771}, \qquad [[[[[[[b,a],b],b],b],c],c],c]_{772},$$
$$[[[[[[[b,a],b],b],c],c],c],c]_{773}, \qquad [[[[[[[b,a],b],c],c],c],c],c]_{774},$$
$$[[[[[[[b,a],c],c],c],c],c],c]_{775}, \qquad [[[[[[[c,a],a],a],a],a],a],a]_{776},$$
$$[[[[[[[c,a],a],a],a],a],a],b]_{777}, \qquad [[[[[[[c,a],a],a],a],a],a],c]_{778},$$
$$[[[[[[[c,a],a],a],a],a],b],b]_{779}, \qquad [[[[[[[c,a],a],a],a],a],b],c]_{780},$$
$$[[[[[[[c,a],a],a],a],a],c],c]_{781}, \qquad [[[[[[[c,a],a],a],a],b],b],b]_{782},$$
$$[[[[[[[c,a],a],a],a],b],b],c]_{783}, \qquad [[[[[[[c,a],a],a],a],b],c],c]_{784},$$
$$[[[[[[[c,a],a],a],a],c],c],c]_{785}, \qquad [[[[[[[c,a],a],a],b],b],b],b]_{786},$$
$$[[[[[[[c,a],a],a],b],b],b],c]_{787}, \qquad [[[[[[[c,a],a],a],b],b],c],c]_{788},$$
$$[[[[[[[c,a],a],a],b],c],c],c]_{789}, \qquad [[[[[[[c,a],a],a],c],c],c],c]_{790},$$
$$[[[[[[[c,a],a],b],b],b],b],b]_{791}, \qquad [[[[[[[c,a],a],b],b],b],b],c]_{792},$$
$$[[[[[[[c,a],a],b],b],b],c],c]_{793}, \qquad [[[[[[[c,a],a],b],b],c],c],c]_{794},$$
$$[[[[[[[c,a],a],b],c],c],c],c]_{795}, \qquad [[[[[[[c,a],a],c],c],c],c],c]_{796},$$
$$[[[[[[[c,a],b],b],b],b],b],b]_{797}, \qquad [[[[[[[c,a],b],b],b],b],b],c]_{798},$$
$$[[[[[[[c,a],b],b],b],b],c],c]_{799}, \qquad [[[[[[[c,a],b],b],b],c],c],c]_{800},$$
$$[[[[[[[c,a],b],b],c],c],c],c]_{801}, \qquad [[[[[[[c,a],b],c],c],c],c],c]_{802},$$
$$[[[[[[[c,a],c],c],c],c],c],c]_{803}, \qquad [[[[[[[c,b],b],b],b],b],b],b]_{804},$$
$$[[[[[[[c,b],b],b],b],b],b],c]_{805}, \qquad [[[[[[[c,b],b],b],b],b],c],c]_{806},$$
$$[[[[[[[c,b],b],b],b],c],c],c]_{807}, \qquad [[[[[[[c,b],b],b],c],c],c],c]_{808},$$
$$[[[[[[[c,b],b],c],c],c],c],c]_{809}, \qquad [[[[[[[c,b],c],c],c],c],c],c]_{810}.$$

D.5 Degree 9

$$[[[[b,a],a],[b,a]],[[c,a],[b,a]]]_{1}, \qquad [[[[b,a],a],[b,a]],[[c,b],[b,a]]]_{2},$$
$$[[[[b,a],a],[b,a]],[[c,b],[c,a]]]_{3}, \qquad [[[[b,a],a],[b,a]],[[[b,a],a],a]]_{4},$$
$$[[[[b,a],a],[b,a]],[[[b,a],a],b]]_{5}, \qquad [[[[b,a],a],[b,a]],[[[b,a],a],c]]_{6},$$
$$[[[[b,a],a],[b,a]],[[[b,a],b],b]]_{7}, \qquad [[[[b,a],a],[b,a]],[[[b,a],b],c]]_{8},$$
$$[[[[b,a],a],[b,a]],[[[b,a],c],c]]_{9}, \qquad [[[[b,a],a],[b,a]],[[[c,a],a],a]]_{10},$$
$$[[[[b,a],a],[b,a]],[[[c,a],a],b]]_{11}, \qquad [[[[b,a],a],[b,a]],[[[c,a],a],c]]_{12},$$
$$[[[[b,a],a],[b,a]],[[[c,a],b],b]]_{13}, \qquad [[[[b,a],a],[b,a]],[[[c,a],b],c]]_{14},$$
$$[[[[b,a],a],[b,a]],[[[c,a],c],c]]_{15}, \qquad [[[[b,a],a],[b,a]],[[[c,b],b],b]]_{16},$$

$[[[[b,a],a],[b,a]],[[[c,b],b],c]]_{17}$, $[[[[b,a],a],[b,a]],[[[c,b],c],c]]_{18}$,
$[[[[b,a],a],[c,a]],[[c,a],[b,a]]]_{19}$, $[[[[b,a],a],[c,a]],[[c,b],[b,a]]]_{20}$,
$[[[[b,a],a],[c,a]],[[c,b],[c,a]]]_{21}$, $[[[[b,a],a],[c,a]],[[[b,a],a],a]]_{22}$,
$[[[[b,a],a],[c,a]],[[[b,a],a],b]]_{23}$, $[[[[b,a],a],[c,a]],[[[b,a],a],c]]_{24}$,
$[[[[b,a],a],[c,a]],[[[b,a],b],b]]_{25}$, $[[[[b,a],a],[c,a]],[[[b,a],b],c]]_{26}$,
$[[[[b,a],a],[c,a]],[[[b,a],c],c]]_{27}$, $[[[[b,a],a],[c,a]],[[[c,a],a],a]]_{28}$,
$[[[[b,a],a],[c,a]],[[[c,a],a],b]]_{29}$, $[[[[b,a],a],[c,a]],[[[c,a],a],c]]_{30}$,
$[[[[b,a],a],[c,a]],[[[c,a],b],b]]_{31}$, $[[[[b,a],a],[c,a]],[[[c,a],b],c]]_{32}$,
$[[[[b,a],a],[c,a]],[[[c,a],c],c]]_{33}$, $[[[[b,a],a],[c,a]],[[[c,b],b],b]]_{34}$,
$[[[[b,a],a],[c,a]],[[[c,b],b],c]]_{35}$, $[[[[b,a],a],[c,a]],[[[c,b],c],c]]_{36}$,
$[[[[b,a],a],[c,b]],[[c,a],[b,a]]]_{37}$, $[[[[b,a],a],[c,b]],[[c,b],[b,a]]]_{38}$,
$[[[[b,a],a],[c,b]],[[c,b],[c,a]]]_{39}$, $[[[[b,a],a],[c,b]],[[[b,a],a],a]]_{40}$,
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$[[[[b,a],a],[c,b]],[[[b,a],c],c]]_{45}$, $[[[[b,a],a],[c,b]],[[[c,a],a],a]]_{46}$,
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$[[[[b,a],a],[c,b]],[[[c,a],c],c]]_{51}$, $[[[[b,a],a],[c,b]],[[[c,b],b],b]]_{52}$,
$[[[[b,a],a],[c,b]],[[[c,b],b],c]]_{53}$, $[[[[b,a],a],[c,b]],[[[c,b],c],c]]_{54}$,
$[[[[b,a],b],[b,a]],[[c,a],[b,a]]]_{55}$, $[[[[b,a],b],[b,a]],[[c,b],[b,a]]]_{56}$,
$[[[[b,a],b],[b,a]],[[c,b],[c,a]]]_{57}$, $[[[[b,a],b],[b,a]],[[[b,a],a],a]]_{58}$,
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$[[[[b,a],b],[b,a]],[[[b,a],b],b]]_{61}$, $[[[[b,a],b],[b,a]],[[[b,a],b],c]]_{62}$,
$[[[[b,a],b],[b,a]],[[[b,a],c],c]]_{63}$, $[[[[b,a],b],[b,a]],[[[c,a],a],a]]_{64}$,
$[[[[b,a],b],[b,a]],[[[c,a],a],b]]_{65}$, $[[[[b,a],b],[b,a]],[[[c,a],a],c]]_{66}$,
$[[[[b,a],b],[b,a]],[[[c,a],b],b]]_{67}$, $[[[[b,a],b],[b,a]],[[[c,a],b],c]]_{68}$,
$[[[[b,a],b],[b,a]],[[[c,a],c],c]]_{69}$, $[[[[b,a],b],[b,a]],[[[c,b],b],b]]_{70}$,
$[[[[b,a],b],[b,a]],[[[c,b],b],c]]_{71}$, $[[[[b,a],b],[b,a]],[[[c,b],c],c]]_{72}$,
$[[[[b,a],b],[c,a]],[[c,a],[b,a]]]_{73}$, $[[[[b,a],b],[c,a]],[[c,b],[b,a]]]_{74}$,
$[[[[b,a],b],[c,a]],[[c,b],[c,a]]]_{75}$, $[[[[b,a],b],[c,a]],[[[b,a],a],a]]_{76}$,
$[[[[b,a],b],[c,a]],[[[b,a],a],b]]_{77}$, $[[[[b,a],b],[c,a]],[[[b,a],a],c]]_{78}$,
$[[[[b,a],b],[c,a]],[[[b,a],b],b]]_{79}$, $[[[[b,a],b],[c,a]],[[[b,a],b],c]]_{80}$,
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$[[[[b,a],b],[c,a]],[[[c,a],c],c]]_{87}$, $[[[[b,a],b],[c,a]],[[[c,b],b],b]]_{88}$,
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$[[[[b,a],b],[c,b]],[[c,b],[c,a]]]_{93}$, $[[[[b,a],b],[c,b]],[[[b,a],a],a]]_{94}$,
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$$[[[[c,a],c],[c,a]],[[[c,a],c],c]]_{303}, \qquad [[[[c,a],c],[c,a]],[[[c,b],b],b]]_{304},$$
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$$[[[[c,a],c],[c,b]],[[c,b],[c,a]]]_{309}, \qquad [[[[c,a],c],[c,b]],[[[b,a],a],a]]_{310},$$
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$[[[[[b,a],a],a],a],[[[b,a],b],b]]_{439}$, $[[[[[b,a],a],a],a],[[[b,a],b],c]]_{440}$,
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$[[[[[b,a],a],a],a],[[[c,a],a],b]]_{443}$, $[[[[[b,a],a],a],a],[[[c,a],a],c]]_{444}$,
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$[[[[[c,a],a],b],b],[[[c,a],a],b]]_{677}$, $[[[[[c,a],a],b],b],[[[c,a],a],c]]_{678}$,
$[[[[[c,a],a],b],b],[[[c,a],b],b]]_{679}$, $[[[[[c,a],a],b],b],[[[c,a],b],c]]_{680}$,
$[[[[[c,a],a],b],b],[[[c,a],c],c]]_{681}$, $[[[[[c,a],a],b],b],[[[c,b],b],b]]_{682}$,

$[[[[[c,a],a],b],b],[[[c,b],b],c]]_{683}$, $[[[[[c,a],a],b],b],[[[c,b],c],c]]_{684}$,
$[[[[[c,a],a],b],c],[[c,a],[b,a]]]_{685}$, $[[[[[c,a],a],b],c],[[c,b],[b,a]]]_{686}$,
$[[[[[c,a],a],b],c],[[c,b],[c,a]]]_{687}$, $[[[[[c,a],a],b],c],[[[b,a],a],a]]_{688}$,
$[[[[[c,a],a],b],c],[[[b,a],a],b]]_{689}$, $[[[[[c,a],a],b],c],[[[b,a],a],c]]_{690}$,
$[[[[[c,a],a],b],c],[[[b,a],b],b]]_{691}$, $[[[[[c,a],a],b],c],[[[b,a],b],c]]_{692}$,
$[[[[[c,a],a],b],c],[[[b,a],c],c]]_{693}$, $[[[[[c,a],a],b],c],[[[c,a],a],a]]_{694}$,
$[[[[[c,a],a],b],c],[[[c,a],a],b]]_{695}$, $[[[[[c,a],a],b],c],[[[c,a],a],c]]_{696}$,
$[[[[[c,a],a],b],c],[[[c,a],b],b]]_{697}$, $[[[[[c,a],a],b],c],[[[c,a],b],c]]_{698}$,
$[[[[[c,a],a],b],c],[[[c,a],c],c]]_{699}$, $[[[[[c,a],a],b],c],[[[c,b],b],b]]_{700}$,
$[[[[[c,a],a],b],c],[[[c,b],b],c]]_{701}$, $[[[[[c,a],a],b],c],[[[c,b],c],c]]_{702}$,
$[[[[[c,a],a],c],c],[[c,a],[b,a]]]_{703}$, $[[[[[c,a],a],c],c],[[c,b],[b,a]]]_{704}$,
$[[[[[c,a],a],c],c],[[c,b],[c,a]]]_{705}$, $[[[[[c,a],a],c],c],[[[b,a],a],a]]_{706}$,
$[[[[[c,a],a],c],c],[[[b,a],a],b]]_{707}$, $[[[[[c,a],a],c],c],[[[b,a],a],c]]_{708}$,
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$[[[[[c,a],b],b],b],[[c,b],[c,a]]]_{723}$, $[[[[[c,a],b],b],b],[[[b,a],a],a]]_{724}$,
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$[[[[[c,a],b],b],c],[[c,b],[c,a]]]_{741}$, $[[[[[c,a],b],b],c],[[[b,a],a],a]]_{742}$,
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$[[[[[c,a],c],c],c],[[[b,a],c],c]]_{783}$, $[[[[[c,a],c],c],c],[[[c,a],a],a]]_{784}$,
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$[[[[[c,b],b],b],b],[[c,a],[b,a]]]_{793}$, $[[[[[c,b],b],b],b],[[c,b],[b,a]]]_{794}$,
$[[[[[c,b],b],b],b],[[c,b],[c,a]]]_{795}$, $[[[[[c,b],b],b],b],[[[b,a],a],a]]_{796}$,
$[[[[[c,b],b],b],b],[[[b,a],a],b]]_{797}$, $[[[[[c,b],b],b],b],[[[b,a],a],c]]_{798}$,
$[[[[[c,b],b],b],b],[[[b,a],b],b]]_{799}$, $[[[[[c,b],b],b],b],[[[b,a],b],c]]_{800}$,
$[[[[[c,b],b],b],b],[[[b,a],c],c]]_{801}$, $[[[[[c,b],b],b],b],[[[c,a],a],a]]_{802}$,
$[[[[[c,b],b],b],b],[[[c,a],a],b]]_{803}$, $[[[[[c,b],b],b],b],[[[c,a],a],c]]_{804}$,
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$[[[[[c,b],b],b],c],[[c,a],[b,a]]]_{811}$, $[[[[[c,b],b],b],c],[[c,b],[b,a]]]_{812}$,
$[[[[[c,b],b],b],c],[[c,b],[c,a]]]_{813}$, $[[[[[c,b],b],b],c],[[[b,a],a],a]]_{814}$,
$[[[[[c,b],b],b],c],[[[b,a],a],b]]_{815}$, $[[[[[c,b],b],b],c],[[[b,a],a],c]]_{816}$,
$[[[[[c,b],b],b],c],[[[b,a],b],b]]_{817}$, $[[[[[c,b],b],b],c],[[[b,a],b],c]]_{818}$,
$[[[[[c,b],b],b],c],[[[b,a],c],c]]_{819}$, $[[[[[c,b],b],b],c],[[[c,a],a],a]]_{820}$,
$[[[[[c,b],b],b],c],[[[c,a],a],b]]_{821}$, $[[[[[c,b],b],b],c],[[[c,a],a],c]]_{822}$,
$[[[[[c,b],b],b],c],[[[c,a],b],b]]_{823}$, $[[[[[c,b],b],b],c],[[[c,a],b],c]]_{824}$,
$[[[[[c,b],b],b],c],[[[c,a],c],c]]_{825}$, $[[[[[c,b],b],b],c],[[[c,b],b],b]]_{826}$,
$[[[[[c,b],b],b],c],[[[c,b],b],c]]_{827}$, $[[[[[c,b],b],b],c],[[[c,b],c],c]]_{828}$,
$[[[[[c,b],b],c],c],[[c,a],[b,a]]]_{829}$, $[[[[[c,b],b],c],c],[[c,b],[b,a]]]_{830}$,

$[[[[[c, b], b], c], c], [[c, b], [c, a]]]_{831}$, $[[[[[c, b], b], c], c], [[[b, a], a], a]]_{832}$,
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$[[[[[c, b], b], c], c], [[[c, a], a], b]]_{839}$, $[[[[[c, b], b], c], c], [[[c, a], a], c]]_{840}$,
$[[[[[c, b], b], c], c], [[[c, a], b], b]]_{841}$, $[[[[[c, b], b], c], c], [[[c, a], b], c]]_{842}$,
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$[[[[[c, b], c], c], c], [[c, a], [b, a]]]_{847}$, $[[[[[c, b], c], c], c], [[c, b], [b, a]]]_{848}$,
$[[[[[c, b], c], c], c], [[c, b], [c, a]]]_{849}$, $[[[[[c, b], c], c], c], [[[b, a], a], a]]_{850}$,
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$[[[[b, a], b], [[b, a], a]], [[b, a], a]]_{865}$, $[[[[b, a], b], [[b, a], a]], [[b, a], b]]_{866}$,
$[[[[b, a], b], [[b, a], a]], [[b, a], c]]_{867}$, $[[[[b, a], b], [[b, a], a]], [[c, a], a]]_{868}$,
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$[[[[b, a], c], [[b, a], a]], [[c, b], b]]_{879}$, $[[[[b, a], c], [[b, a], a]], [[c, b], c]]_{880}$,
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$[[[[b, a], c], [[b, a], b]], [[c, b], c]]_{887}$, $[[[[c, a], a], [[b, a], a]], [[b, a], a]]_{888}$,
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$[[[[c, a], a], [[b, a], b]], [[b, a], c]]_{897}$, $[[[[c, a], a], [[b, a], b]], [[c, a], a]]_{898}$,
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$[[[[c,a],a],[[b,a],c]],[[c,a],b]]_{905}$, $[[[[c,a],a],[[b,a],c]],[[c,a],c]]_{906}$,
$[[[[c,a],a],[[b,a],c]],[[c,b],b]]_{907}$, $[[[[c,a],a],[[b,a],c]],[[c,b],c]]_{908}$,
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$[[[[c,a],c],[[c,a],b]],[[c,b],b]]_{963}$, $[[[[c,a],c],[[c,a],b]],[[c,b],c]]_{964}$,
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$$[[[[[c,a],a],a],[c,b]],[[b,a],c]]_{1259},\qquad [[[[[c,a],a],a],[c,b]],[[c,a],a]]_{1260},$$
$$[[[[[c,a],a],a],[c,b]],[[c,a],b]]_{1261},\qquad [[[[[c,a],a],a],[c,b]],[[c,a],c]]_{1262},$$
$$[[[[[c,a],a],a],[c,b]],[[c,b],b]]_{1263},\qquad [[[[[c,a],a],a],[c,b]],[[c,b],c]]_{1264},$$
$$[[[[[c,a],a],b],[b,a]],[[b,a],a]]_{1265},\qquad [[[[[c,a],a],b],[b,a]],[[b,a],b]]_{1266},$$
$$[[[[[c,a],a],b],[b,a]],[[b,a],c]]_{1267},\qquad [[[[[c,a],a],b],[b,a]],[[c,a],a]]_{1268},$$
$$[[[[[c,a],a],b],[b,a]],[[c,a],b]]_{1269},\qquad [[[[[c,a],a],b],[b,a]],[[c,a],c]]_{1270},$$
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$[[[[[[[b,a],a],b],c],c],c],c],c]_{2131}$, $[[[[[[[b,a],a],c],c],c],c],c],c]_{2132}$,
$[[[[[[[b,a],b],b],b],b],b],b],b]_{2133}$, $[[[[[[[b,a],b],b],b],b],b],b],c]_{2134}$,
$[[[[[[[b,a],b],b],b],b],b],c],c]_{2135}$, $[[[[[[[b,a],b],b],b],b],c],c],c]_{2136}$,
$[[[[[[[b,a],b],b],b],c],c],c],c]_{2137}$, $[[[[[[[b,a],b],b],c],c],c],c],c]_{2138}$,
$[[[[[[[b,a],b],c],c],c],c],c],c]_{2139}$, $[[[[[[[b,a],c],c],c],c],c],c],c]_{2140}$,
$[[[[[[[c,a],a],a],a],a],a],a],a]_{2141}$, $[[[[[[[c,a],a],a],a],a],a],a],b]_{2142}$,
$[[[[[[[c,a],a],a],a],a],a],a],c]_{2143}$, $[[[[[[[c,a],a],a],a],a],a],b],b]_{2144}$,
$[[[[[[[c,a],a],a],a],a],a],b],c]_{2145}$, $[[[[[[[c,a],a],a],a],a],a],c],c]_{2146}$,
$[[[[[[[c,a],a],a],a],a],b],b],b]_{2147}$, $[[[[[[[c,a],a],a],a],a],b],b],c]_{2148}$,
$[[[[[[[c,a],a],a],a],a],b],c],c]_{2149}$, $[[[[[[[c,a],a],a],a],a],c],c],c]_{2150}$,
$[[[[[[[c,a],a],a],a],b],b],b],b]_{2151}$, $[[[[[[[c,a],a],a],a],b],b],b],c]_{2152}$,
$[[[[[[[c,a],a],a],a],b],b],c],c]_{2153}$, $[[[[[[[c,a],a],a],a],b],c],c],c]_{2154}$,
$[[[[[[[c,a],a],a],a],c],c],c],c]_{2155}$, $[[[[[[[c,a],a],a],b],b],b],b],b]_{2156}$,
$[[[[[[[c,a],a],a],b],b],b],b],c]_{2157}$, $[[[[[[[c,a],a],a],b],b],b],c],c]_{2158}$,
$[[[[[[[c,a],a],a],b],b],c],c],c]_{2159}$, $[[[[[[[c,a],a],a],b],c],c],c],c]_{2160}$,
$[[[[[[[c,a],a],a],c],c],c],c],c]_{2161}$, $[[[[[[[c,a],a],b],b],b],b],b],b]_{2162}$,

$[[[[[[[c,a],a],b],b],b],b],b],c]_{2163}$, $[[[[[[[c,a],a],b],b],b],b],c],c]_{2164}$,
$[[[[[[[c,a],a],b],b],b],c],c],c]_{2165}$, $[[[[[[[c,a],a],b],b],c],c],c],c]_{2166}$,
$[[[[[[[c,a],a],b],c],c],c],c],c]_{2167}$, $[[[[[[[c,a],a],c],c],c],c],c],c]_{2168}$,
$[[[[[[[c,a],b],b],b],b],b],b],b]_{2169}$, $[[[[[[[c,a],b],b],b],b],b],b],c]_{2170}$,
$[[[[[[[c,a],b],b],b],b],b],c],c]_{2171}$, $[[[[[[[c,a],b],b],b],b],c],c],c]_{2172}$,
$[[[[[[[c,a],b],b],b],c],c],c],c]_{2173}$, $[[[[[[[c,a],b],b],c],c],c],c],c]_{2174}$,
$[[[[[[[c,a],b],c],c],c],c],c],c]_{2175}$, $[[[[[[[c,a],c],c],c],c],c],c],c]_{2176}$,
$[[[[[[[c,b],b],b],b],b],b],b],b]_{2177}$, $[[[[[[[c,b],b],b],b],b],b],b],c]_{2178}$,
$[[[[[[[c,b],b],b],b],b],b],c],c]_{2179}$, $[[[[[[[c,b],b],b],b],b],c],c],c]_{2180}$,
$[[[[[[[c,b],b],b],b],c],c],c],c]_{2181}$, $[[[[[[[c,b],b],b],c],c],c],c],c]_{2182}$,
$[[[[[[[c,b],b],c],c],c],c],c],c]_{2183}$, $[[[[[[[c,b],c],c],c],c],c],c],c]_{2184}$.

Appendix E

Action equations

E.1 Degree 12 in the natural representation of sl(2)

$x.X_{12}^{2} = X_{12}^{1},$ $x.X_{12}^{5} = 2X_{12}^{3},$
$x.X_{12}^{14} = X_{12}^{10} + 2X_{12}^{13},$ $x.X_{12}^{16} = 2X_{12}^{11} + X_{12}^{1},$
$x.X_{12}^{18} = 2X_{12}^{13} - X_{12}^{3} + X_{12}^{17},$ $x.X_{12}^{23} = 3X_{12}^{17} + 2X_{12}^{1} + 3X_{12}^{3},$
$x.X_{12}^{27} = 3X_{12}^{21} + 2X_{12}^{11} - 3X_{12}^{13} + X_{12}^{26},$
$x.X_{12}^{33} = 4X_{12}^{26} - 3X_{12}^{10},$ $x.X_{12}^{42} = 3X_{12}^{41} + X_{12}^{38},$
$x.X_{12}^{44} = X_{12}^{38} + X_{12}^{43},$ $x.X_{12}^{47} = X_{12}^{41} + 2X_{12}^{46} + X_{12}^{43},$
$x.X_{12}^{50} = X_{12}^{38} + X_{12}^{49},$ $x.X_{12}^{53} = X_{12}^{41} + 2X_{12}^{52} + X_{12}^{49},$
$x.X_{12}^{55} = X_{12}^{43} + X_{12}^{49},$ $x.X_{12}^{58} = X_{12}^{46} + X_{12}^{52} + X_{12}^{57},$
$x.X_{12}^{61} = 2X_{12}^{49},$ $x.X_{12}^{64} = 2X_{12}^{52} + X_{12}^{63},$
$x.X_{12}^{69} = 2X_{12}^{57} + X_{12}^{63},$ $x.X_{12}^{74} = X_{12}^{73},$
$x.X_{12}^{77} = 2X_{12}^{76} + X_{12}^{73},$ $x.X_{12}^{79} = X_{12}^{73},$
$x.X_{12}^{82} = X_{12}^{76} + X_{12}^{81},$ $x.X_{12}^{90} = 3X_{12}^{89} + X_{12}^{86},$
$x.X_{12}^{92} = X_{12}^{86} + X_{12}^{91},$ $x.X_{12}^{95} = X_{12}^{89} + 2X_{12}^{94} + X_{12}^{91},$
$x.X_{12}^{97} = 2X_{12}^{91} + X_{12}^{73},$ $x.X_{12}^{100} = 2X_{12}^{94} + X_{12}^{76} + X_{12}^{99},$
$x.X_{12}^{105} = 3X_{12}^{99} + X_{12}^{81},$ $x.X_{12}^{120} = X_{12}^{114} + 3X_{12}^{119} + X_{12}^{116},$
$x.X_{12}^{122} = 2X_{12}^{116} + X_{12}^{86} + X_{12}^{121},$
$x.X_{12}^{125} = 2X_{12}^{119} + X_{12}^{89} + 2X_{12}^{124} + X_{12}^{121},$
$x.X_{12}^{127} = 3X_{12}^{121} + 2X_{12}^{43} + 3X_{12}^{91},$
$x.X_{12}^{130} = 3X_{12}^{124} + 2X_{12}^{46} + 3X_{12}^{94} + X_{12}^{129},$
$x.X_{12}^{135} = 4X_{12}^{129} + 8X_{12}^{57} - 3X_{12}^{63} + 6X_{12}^{99},$
$x.X_{12}^{147} = 2X_{12}^{146},$ $x.X_{12}^{149} = 2X_{12}^{146} + X_{12}^{148},$
$x.X_{12}^{155} = 2X_{12}^{154},$ $x.X_{12}^{157} = X_{12}^{154} + X_{12}^{156},$
$x.X_{12}^{160} = X_{12}^{154} + X_{12}^{159},$ $x.X_{12}^{162} = X_{12}^{156} + X_{12}^{159},$
$x.X_{12}^{170} = X_{12}^{167} + 2X_{12}^{169},$ $x.X_{12}^{173} = X_{12}^{167} + 2X_{12}^{172},$
$x.X_{12}^{175} = X_{12}^{169} + X_{12}^{172} + X_{12}^{174},$ $x.X_{12}^{178} = 2X_{12}^{172} + X_{12}^{154} + X_{12}^{177},$
$x.X_{12}^{180} = 2X_{12}^{174} + X_{12}^{156} + X_{12}^{177},$ $x.X_{12}^{183} = 3X_{12}^{177} + X_{12}^{159},$
$x.X_{12}^{191} = 2X_{12}^{190},$ $x.X_{12}^{193} = X_{12}^{190} + X_{12}^{192},$
$x.X_{12}^{195} = 2X_{12}^{192},$ $x.X_{12}^{203} = X_{12}^{200} + 2X_{12}^{202},$
$x.X_{12}^{205} = 2X_{12}^{202} + X_{12}^{190} + X_{12}^{204},$
$x.X_{12}^{207} = 3X_{12}^{204} + 2X_{12}^{156} - 2X_{12}^{159} + 3X_{12}^{192},$
$x.X_{12}^{221} = 2X_{12}^{218} + X_{12}^{200} + 2X_{12}^{220},$
$x.X_{12}^{223} = 3X_{12}^{220} + 2X_{12}^{169} + 3X_{12}^{202} + X_{12}^{222},$
$x.X_{12}^{225} = 4X_{12}^{222} - 3X_{12}^{148} + 8X_{12}^{174} + 6X_{12}^{204},$

$x.X_{12}^{235} = X_{12}^{234}$, $x.X_{12}^{239} = 2X_{12}^{238} + X_{12}^{1}$,
$x.X_{12}^{241} = X_{12}^{238} + X_{12}^{240}$, $x.X_{12}^{243} = 2X_{12}^{240}$,
$x.X_{12}^{251} = X_{12}^{248} + 2X_{12}^{250} + X_{12}^{11}$, $x.X_{12}^{253} = 2X_{12}^{250} + X_{12}^{238} + X_{12}^{252}$,
$x.X_{12}^{255} = 3X_{12}^{252} - 2X_{12}^{234} + 3X_{12}^{240}$, $x.X_{12}^{262} = X_{12}^{261}$,
$x.X_{12}^{263} = X_{12}^{261}$, $x.X_{12}^{268} = X_{12}^{266} + X_{12}^{267}$,
$x.X_{12}^{269} = 2X_{12}^{267} + X_{12}^{261}$, $x.X_{12}^{278} = 2X_{12}^{276} + X_{12}^{266} + X_{12}^{277}$,
$x.X_{12}^{279} = 3X_{12}^{277} + 2X_{12}^{38} - 2X_{12}^{159} + 2X_{12}^{1} + 2X_{12}^{238} + 3X_{12}^{267}$,
$x.X_{12}^{292} = 3X_{12}^{290} + 2X_{12}^{248} + 3X_{12}^{276} + X_{12}^{291}$,
$x.X_{12}^{293} = 4X_{12}^{291} - 3X_{12}^{63} + 3X_{12}^{167} + 8X_{12}^{11} + 8X_{12}^{250} + 6X_{12}^{277}$,
$x.X_{12}^{302} = X_{12}^{301}$, $x.X_{12}^{306} = 2X_{12}^{305} + X_{12}^{301}$,
$x.X_{12}^{312} = 3X_{12}^{311} - 2X_{12}^{81} + 4X_{12}^{86} + 2X_{12}^{266} + 3X_{12}^{305}$,
$x.X_{12}^{320} = 4X_{12}^{319} + 3X_{12}^{10} + 3X_{12}^{200} + 8X_{12}^{116} + 8X_{12}^{276} + 6X_{12}^{311}$,
$x.X_{12}^{330} = 5X_{12}^{329} + 4X_{12}^{114} + 15X_{12}^{218} + 20X_{12}^{290} + 10X_{12}^{319}$.

E.2 Degree 14 in the natural representation of sl(2)

$x.X_{14}^{5} = X_{14}^{3} + X_{14}^{4}$, $x.X_{14}^{6} = -X_{14}^{3} + X_{14}^{4}$,
$x.X_{14}^{8} = 2X_{14}^{3} + X_{14}^{7}$, $x.X_{14}^{9} = X_{14}^{7}$,
$x.X_{14}^{11} = 2X_{14}^{4} + X_{14}^{7}$, $x.X_{14}^{19} = X_{14}^{17} + X_{14}^{18}$,
$x.X_{14}^{20} = 2X_{14}^{18}$, $x.X_{14}^{23} = X_{14}^{17} + X_{14}^{22}$,
$x.X_{14}^{24} = X_{14}^{18} + X_{14}^{22}$, $x.X_{14}^{28} = 0$,
$x.X_{14}^{34} = 2X_{14}^{32} + X_{14}^{33}$, $x.X_{14}^{40} = X_{14}^{32} + X_{14}^{38} + X_{14}^{39}$,
$x.X_{14}^{41} = X_{14}^{33} + 2X_{14}^{39}$, $x.X_{14}^{44} = X_{14}^{36} + X_{14}^{43}$,
$x.X_{14}^{47} = 2X_{14}^{38} + X_{14}^{17} + X_{14}^{46}$, $x.X_{14}^{48} = 2X_{14}^{39} + X_{14}^{18} + X_{14}^{46}$,
$x.X_{14}^{52} = 2X_{14}^{43}$, $x.X_{14}^{55} = -X_{14}^{43} + X_{14}^{54}$,
$x.X_{14}^{56} = 3X_{14}^{46} + X_{14}^{22}$, $x.X_{14}^{64} = 3X_{14}^{54} - X_{14}^{36}$,
$x.X_{14}^{84} = X_{14}^{72} + 2X_{14}^{82} + X_{14}^{83}$,
$x.X_{14}^{90} = X_{14}^{78} + 3X_{14}^{89} + X_{14}^{86}$,
$x.X_{14}^{95} = 2X_{14}^{82} + X_{14}^{32} + X_{14}^{93} + X_{14}^{94}$,
$x.X_{14}^{96} = 2X_{14}^{83} + X_{14}^{33} + 2X_{14}^{94}$, $x.X_{14}^{99} = 2X_{14}^{86} + X_{14}^{36} + X_{14}^{98}$,
$x.X_{14}^{102} = 2X_{14}^{89} - X_{14}^{54} + 2X_{14}^{101} + X_{14}^{98}$,
$x.X_{14}^{107} = 3X_{14}^{93} + 3X_{14}^{38} + X_{14}^{106}$,
$x.X_{14}^{108} = 3X_{14}^{94} - 2X_{14}^{3} + 3X_{14}^{39} + X_{14}^{106}$,
$x.X_{14}^{112} = 3X_{14}^{98} - 2X_{14}^{17} + 3X_{14}^{43}$, $x.X_{14}^{115} = 3X_{14}^{101} - 2X_{14}^{38} + X_{14}^{114}$,
$x.X_{14}^{120} = -2X_{14}^{93} - 3X_{14}^{101} + 2X_{14}^{119} + X_{14}^{114}$,
$x.X_{14}^{121} = 4X_{14}^{106} + 8X_{14}^{4} - 3X_{14}^{7} + 6X_{14}^{46}$,
$x.X_{14}^{129} = 4X_{14}^{114} - 8X_{14}^{32} + 3X_{14}^{33} + 6X_{14}^{54}$,
$x.X_{14}^{134} = 4X_{14}^{119} - 8X_{14}^{82} + 3X_{14}^{83} - 6X_{14}^{89} + X_{14}^{133}$,
$x.X_{14}^{149} = 5X_{14}^{133} - 5X_{14}^{72} - 6X_{14}^{78}$, $x.X_{14}^{157} = 2X_{14}^{156}$,
$x.X_{14}^{161} = 3X_{14}^{160} - 2X_{14}^{154} + 3X_{14}^{156}$, $x.X_{14}^{163} = 2X_{14}^{154}$,
$x.X_{14}^{165} = 2X_{14}^{156} + X_{14}^{164}$, $x.X_{14}^{169} = 2X_{14}^{160} + 2X_{14}^{168} + X_{14}^{164}$,
$x.X_{14}^{173} = X_{14}^{164}$, $x.X_{14}^{177} = X_{14}^{168} + X_{14}^{176}$,
$x.X_{14}^{184} = 2X_{14}^{183}$, $x.X_{14}^{188} = 3X_{14}^{187} - 2X_{14}^{181} + 3X_{14}^{183}$,

$x.X_{14}^{190} = X_{14}^{181}$, $x.X_{14}^{192} = X_{14}^{183} + X_{14}^{191}$,
$x.X_{14}^{196} = X_{14}^{187} + 2X_{14}^{195} + X_{14}^{191}$, $x.X_{14}^{199} = X_{14}^{181}$,
$x.X_{14}^{201} = X_{14}^{183} + X_{14}^{200}$, $x.X_{14}^{205} = X_{14}^{187} + 2X_{14}^{204} + X_{14}^{200}$,
$x.X_{14}^{209} = X_{14}^{191} + X_{14}^{200}$, $x.X_{14}^{213} = X_{14}^{195} + X_{14}^{204} + X_{14}^{212}$,
$x.X_{14}^{225} = 4X_{14}^{224} + 3X_{14}^{220}$, $x.X_{14}^{229} = X_{14}^{220} + 2X_{14}^{228}$,
$x.X_{14}^{233} = X_{14}^{224} + 3X_{14}^{232} - 2X_{14}^{226} + 3X_{14}^{228}$,
$x.X_{14}^{238} = X_{14}^{220} + 2X_{14}^{237}$,
$x.X_{14}^{242} = X_{14}^{224} + 3X_{14}^{241} - 2X_{14}^{235} + 3X_{14}^{237}$,
$x.X_{14}^{244} = X_{14}^{226} + X_{14}^{235}$, $x.X_{14}^{246} = X_{14}^{228} + X_{14}^{237} + X_{14}^{245}$,
$x.X_{14}^{250} = X_{14}^{232} + X_{14}^{241} + 2X_{14}^{249} + X_{14}^{245}$,
$x.X_{14}^{253} = 2X_{14}^{235} + X_{14}^{181}$, $x.X_{14}^{255} = 2X_{14}^{237} + X_{14}^{183} + X_{14}^{254}$,
$x.X_{14}^{259} = 2X_{14}^{241} + X_{14}^{187} + 2X_{14}^{258} + X_{14}^{254}$,
$x.X_{14}^{263} = 2X_{14}^{245} + X_{14}^{191} + X_{14}^{254}$,
$x.X_{14}^{267} = 2X_{14}^{249} + X_{14}^{195} + X_{14}^{258} + X_{14}^{266}$,
$x.X_{14}^{272} = 3X_{14}^{254} + X_{14}^{200}$, $x.X_{14}^{276} = 3X_{14}^{258} + X_{14}^{204} + X_{14}^{275}$,
$x.X_{14}^{284} = 3X_{14}^{266} + X_{14}^{212} + X_{14}^{275}$, $x.X_{14}^{292} = 2X_{14}^{291}$,
$x.X_{14}^{296} = 3X_{14}^{295} - 2X_{14}^{289} + 3X_{14}^{291}$, $x.X_{14}^{298} = X_{14}^{289}$,
$x.X_{14}^{300} = X_{14}^{291} + X_{14}^{299}$, $x.X_{14}^{304} = X_{14}^{295} + 2X_{14}^{303} + X_{14}^{299}$,
$x.X_{14}^{308} = 2X_{14}^{299}$, $x.X_{14}^{312} = 2X_{14}^{303} + X_{14}^{311}$,
$x.X_{14}^{324} = 4X_{14}^{323} + 3X_{14}^{319}$, $x.X_{14}^{328} = X_{14}^{319} + 2X_{14}^{327}$,
$x.X_{14}^{332} = X_{14}^{323} + 3X_{14}^{331} - 2X_{14}^{325} + 3X_{14}^{327}$,
$x.X_{14}^{334} = 2X_{14}^{325} + X_{14}^{289}$, $x.X_{14}^{336} = 2X_{14}^{327} + X_{14}^{291} + X_{14}^{335}$,
$x.X_{14}^{340} = 2X_{14}^{331} + X_{14}^{295} + 2X_{14}^{339} + X_{14}^{335}$,
$x.X_{14}^{344} = 3X_{14}^{335} + 2X_{14}^{191} - 2X_{14}^{200} + 3X_{14}^{299}$,
$x.X_{14}^{348} = 3X_{14}^{339} + 2X_{14}^{195} - 2X_{14}^{204} + 3X_{14}^{303} + X_{14}^{347}$,
$x.X_{14}^{356} = 4X_{14}^{347} + 3X_{14}^{311}$, $x.X_{14}^{378} = X_{14}^{369} + 4X_{14}^{377} + 3X_{14}^{373}$,
$x.X_{14}^{382} = 2X_{14}^{373} + X_{14}^{319} + 2X_{14}^{381}$,
$x.X_{14}^{386} = 2X_{14}^{377} + X_{14}^{323} + 3X_{14}^{385} - 2X_{14}^{379} + 3X_{14}^{381}$,
$x.X_{14}^{388} = 3X_{14}^{379} + 2X_{14}^{226} + 3X_{14}^{325}$,
$x.X_{14}^{390} = 3X_{14}^{381} + 2X_{14}^{228} + 3X_{14}^{327} + X_{14}^{389}$,
$x.X_{14}^{394} = 3X_{14}^{385} + 2X_{14}^{232} + 3X_{14}^{331} + 2X_{14}^{393} + X_{14}^{389}$,
$x.X_{14}^{398} = 4X_{14}^{389} - 3X_{14}^{164} + 8X_{14}^{245} + 6X_{14}^{335}$,
$x.X_{14}^{402} = 4X_{14}^{393} - 3X_{14}^{168} + 8X_{14}^{249} + 6X_{14}^{339} + X_{14}^{401}$,
$x.X_{14}^{410} = 5X_{14}^{401} - 15X_{14}^{176} + 20X_{14}^{266} - 4X_{14}^{275} + 10X_{14}^{347}$,
$x.X_{14}^{429} = 3X_{14}^{428} + X_{14}^{425}$, $x.X_{14}^{431} = X_{14}^{425} + X_{14}^{430}$,
$x.X_{14}^{434} = X_{14}^{428} + 2X_{14}^{433} + X_{14}^{430}$, $x.X_{14}^{436} = 2X_{14}^{430}$,
$x.X_{14}^{439} = 2X_{14}^{433} + X_{14}^{438}$, $x.X_{14}^{443} = X_{14}^{425} + X_{14}^{442}$,
$x.X_{14}^{446} = X_{14}^{428} + 2X_{14}^{445} + X_{14}^{442}$, $x.X_{14}^{448} = X_{14}^{430} + X_{14}^{442}$,
$x.X_{14}^{451} = X_{14}^{433} + X_{14}^{445} + X_{14}^{450}$, $x.X_{14}^{456} = X_{14}^{438} + 2X_{14}^{450}$,
$x.X_{14}^{471} = X_{14}^{465} + 3X_{14}^{470} + X_{14}^{467}$, $x.X_{14}^{473} = 2X_{14}^{467} + X_{14}^{472}$,
$x.X_{14}^{476} = 2X_{14}^{470} + 2X_{14}^{475} + X_{14}^{472}$, $x.X_{14}^{483} = X_{14}^{465} + 3X_{14}^{482} + X_{14}^{479}$,
$x.X_{14}^{485} = X_{14}^{467} + X_{14}^{479} + X_{14}^{484}$,
$x.X_{14}^{488} = X_{14}^{470} + X_{14}^{482} + 2X_{14}^{487} + X_{14}^{484}$,
$x.X_{14}^{490} = X_{14}^{472} + 2X_{14}^{484}$, $x.X_{14}^{493} = X_{14}^{475} + 2X_{14}^{487} + X_{14}^{492}$,
$x.X_{14}^{497} = 2X_{14}^{479} + X_{14}^{425} + X_{14}^{496}$,

$x.X_{14}^{500} = 2X_{14}^{482} + X_{14}^{428} + 2X_{14}^{499} + X_{14}^{496}$,
$x.X_{14}^{502} = 2X_{14}^{484} + X_{14}^{430} + X_{14}^{496}$,
$x.X_{14}^{505} = 2X_{14}^{487} + X_{14}^{433} + X_{14}^{499} + X_{14}^{504}$,
$x.X_{14}^{510} = 2X_{14}^{492} + X_{14}^{438} + 2X_{14}^{504}$,
$x.X_{14}^{514} = 3X_{14}^{496} + X_{14}^{442}$,
$x.X_{14}^{517} = 3X_{14}^{499} + X_{14}^{445} + X_{14}^{516}$,
$x.X_{14}^{522} = 3X_{14}^{504} + X_{14}^{450} + X_{14}^{516}$,
$x.X_{14}^{533} = X_{14}^{532}$,
$x.X_{14}^{536} = 2X_{14}^{535} + X_{14}^{532}$,
$x.X_{14}^{538} = X_{14}^{532}$,
$x.X_{14}^{541} = X_{14}^{535} + X_{14}^{540}$,
$x.X_{14}^{549} = 3X_{14}^{548} + X_{14}^{545}$,
$x.X_{14}^{551} = X_{14}^{545} + X_{14}^{550}$,
$x.X_{14}^{554} = X_{14}^{548} + 2X_{14}^{553} + X_{14}^{550}$,
$x.X_{14}^{557} = X_{14}^{545} + X_{14}^{556}$,
$x.X_{14}^{560} = X_{14}^{548} + 2X_{14}^{559} + X_{14}^{556}$,
$x.X_{14}^{562} = X_{14}^{550} + X_{14}^{556}$,
$x.X_{14}^{565} = X_{14}^{553} + X_{14}^{559} + X_{14}^{564}$,
$x.X_{14}^{568} = 2X_{14}^{556}$,
$x.X_{14}^{571} = 2X_{14}^{559} + X_{14}^{570}$,
$x.X_{14}^{576} = 2X_{14}^{564} + X_{14}^{570}$,
$x.X_{14}^{591} = X_{14}^{585} + 3X_{14}^{590} + X_{14}^{587}$,
$x.X_{14}^{597} = X_{14}^{585} + 3X_{14}^{596} + X_{14}^{593}$,
$x.X_{14}^{599} = X_{14}^{587} + X_{14}^{593} + X_{14}^{598}$,
$x.X_{14}^{602} = X_{14}^{590} + X_{14}^{596} + 2X_{14}^{601} + X_{14}^{598}$,
$x.X_{14}^{605} = 2X_{14}^{593} + X_{14}^{545} + X_{14}^{604}$,
$x.X_{14}^{608} = 2X_{14}^{596} + X_{14}^{548} + 2X_{14}^{607} + X_{14}^{604}$,
$x.X_{14}^{610} = 2X_{14}^{598} + X_{14}^{550} + X_{14}^{604}$,
$x.X_{14}^{613} = 2X_{14}^{601} + X_{14}^{553} + X_{14}^{607} + X_{14}^{612}$,
$x.X_{14}^{616} = 3X_{14}^{604} - 2X_{14}^{532} + 3X_{14}^{556}$,
$x.X_{14}^{619} = 3X_{14}^{607} - 2X_{14}^{535} + 3X_{14}^{559} + X_{14}^{618}$,
$x.X_{14}^{624} = 3X_{14}^{612} - 2X_{14}^{540} + 3X_{14}^{564} + X_{14}^{618}$,
$x.X_{14}^{630} = 4X_{14}^{618} + 3X_{14}^{570}$,
$x.X_{14}^{641} = X_{14}^{640}$,
$x.X_{14}^{644} = 2X_{14}^{643} + X_{14}^{640}$,
$x.X_{14}^{646} = X_{14}^{640}$,
$x.X_{14}^{649} = X_{14}^{643} + X_{14}^{648}$,
$x.X_{14}^{657} = 3X_{14}^{656} + X_{14}^{653}$,
$x.X_{14}^{659} = X_{14}^{653} + X_{14}^{658}$,
$x.X_{14}^{662} = X_{14}^{656} + 2X_{14}^{661} + X_{14}^{658}$,
$x.X_{14}^{664} = 2X_{14}^{658} + X_{14}^{640}$,
$x.X_{14}^{667} = 2X_{14}^{661} + X_{14}^{643} + X_{14}^{666}$,
$x.X_{14}^{672} = 3X_{14}^{666} + X_{14}^{648}$,
$x.X_{14}^{687} = X_{14}^{681} + 3X_{14}^{686} + X_{14}^{683}$,
$x.X_{14}^{689} = 2X_{14}^{683} + X_{14}^{653} + X_{14}^{688}$,
$x.X_{14}^{692} = 2X_{14}^{686} + X_{14}^{656} + 2X_{14}^{691} + X_{14}^{688}$,
$x.X_{14}^{694} = 3X_{14}^{688} - 2X_{14}^{442} + 2X_{14}^{550} + 3X_{14}^{658}$,
$x.X_{14}^{697} = 3X_{14}^{691} - 2X_{14}^{445} + 2X_{14}^{553} + 3X_{14}^{661} + X_{14}^{696}$,
$x.X_{14}^{702} = 4X_{14}^{696} - 8X_{14}^{450} + 8X_{14}^{564} + 3X_{14}^{438} - 3X_{14}^{570} + 6X_{14}^{666}$,
$x.X_{14}^{729} = 2X_{14}^{723} + X_{14}^{681} + 3X_{14}^{728} + X_{14}^{725}$,
$x.X_{14}^{731} = 3X_{14}^{725} + 2X_{14}^{587} + 3X_{14}^{683} + X_{14}^{730}$,
$x.X_{14}^{734} = 3X_{14}^{728} + 2X_{14}^{590} + 3X_{14}^{686} + 2X_{14}^{733} + X_{14}^{730}$,
$x.X_{14}^{736} = 4X_{14}^{730} + 3X_{14}^{472} + 8X_{14}^{598} + 6X_{14}^{688}$,
$x.X_{14}^{739} = 4X_{14}^{733} + 3X_{14}^{475} + 8X_{14}^{601} + 6X_{14}^{691} + X_{14}^{738}$,
$x.X_{14}^{744} = 5X_{14}^{738} + 15X_{14}^{492} - 4X_{14}^{516} + 20X_{14}^{612} + 10X_{14}^{696}$,
$x.X_{14}^{762} = 2X_{14}^{761}$,
$x.X_{14}^{763} = 2X_{14}^{761} - X_{14}^{154}$,
$x.X_{14}^{770} = X_{14}^{768} + 2X_{14}^{769}$,
$x.X_{14}^{774} = X_{14}^{768} + 2X_{14}^{773}$,
$x.X_{14}^{775} = X_{14}^{769} + 2X_{14}^{773} - X_{14}^{156}$,
$x.X_{14}^{779} = 2X_{14}^{773} + X_{14}^{778}$,
$x.X_{14}^{789} = 4X_{14}^{788} - 2X_{14}^{176}$,
$x.X_{14}^{794} = X_{14}^{788} + X_{14}^{792} + 2X_{14}^{793}$,
$x.X_{14}^{798} = 2X_{14}^{792} + X_{14}^{768} + 2X_{14}^{797}$,
$x.X_{14}^{799} = 2X_{14}^{793} + X_{14}^{769} + 2X_{14}^{797} - X_{14}^{160}$,

$x.X_{14}^{803} = 3X_{14}^{797} - 2X_{14}^{761} + 3X_{14}^{773} + X_{14}^{802}$,
$x.X_{14}^{808} = 4X_{14}^{802} + 3X_{14}^{778}$, $x.X_{14}^{819} = X_{14}^{816} + 2X_{14}^{818}$,
$x.X_{14}^{821} = 2X_{14}^{818} - X_{14}^{761} + X_{14}^{820}$, $x.X_{14}^{825} = X_{14}^{816} + 2X_{14}^{824}$,
$x.X_{14}^{827} = X_{14}^{818} + X_{14}^{824} + X_{14}^{826}$,
$x.X_{14}^{829} = X_{14}^{820} + 2X_{14}^{826} + X_{14}^{154} - X_{14}^{761}$,
$x.X_{14}^{833} = 2X_{14}^{824} + X_{14}^{832}$, $x.X_{14}^{835} = 2X_{14}^{826} + X_{14}^{832}$,
$x.X_{14}^{843} = 2X_{14}^{842}$, $x.X_{14}^{845} = X_{14}^{842} + X_{14}^{844}$,
$x.X_{14}^{848} = X_{14}^{842} + X_{14}^{847}$, $x.X_{14}^{850} = X_{14}^{844} + X_{14}^{847}$,
$x.X_{14}^{858} = X_{14}^{855} + 2X_{14}^{857}$, $x.X_{14}^{861} = X_{14}^{855} + 2X_{14}^{860}$,
$x.X_{14}^{863} = X_{14}^{857} + X_{14}^{860} + X_{14}^{862}$, $x.X_{14}^{866} = 2X_{14}^{860} + X_{14}^{842} + X_{14}^{865}$,
$x.X_{14}^{868} = 2X_{14}^{862} + X_{14}^{844} + X_{14}^{865}$, $x.X_{14}^{871} = 3X_{14}^{865} + X_{14}^{847}$,
$x.X_{14}^{888} = X_{14}^{882} + X_{14}^{885} + 2X_{14}^{887}$, $x.X_{14}^{891} = 2X_{14}^{885} + X_{14}^{855} + 2X_{14}^{890}$,
$x.X_{14}^{893} = 2X_{14}^{887} + X_{14}^{857} + X_{14}^{890} + X_{14}^{892}$,
$x.X_{14}^{896} = 3X_{14}^{890} - 2X_{14}^{761} + 2X_{14}^{818} + 3X_{14}^{860} + X_{14}^{895}$,
$x.X_{14}^{898} = 3X_{14}^{892} + 2X_{14}^{820} + 3X_{14}^{862} + X_{14}^{895}$,
$x.X_{14}^{901} = 4X_{14}^{895} + 8X_{14}^{154} - 8X_{14}^{761} + 8X_{14}^{826} - 3X_{14}^{832} + 6X_{14}^{865}$,
$x.X_{14}^{915} = 2X_{14}^{914}$, $x.X_{14}^{917} = X_{14}^{914} + X_{14}^{916}$,
$x.X_{14}^{919} = 2X_{14}^{916}$, $x.X_{14}^{927} = X_{14}^{924} + 2X_{14}^{926}$,
$x.X_{14}^{929} = 2X_{14}^{926} + X_{14}^{914} + X_{14}^{928}$,
$x.X_{14}^{931} = 3X_{14}^{928} + 4X_{14}^{425} - 4X_{14}^{442} + 2X_{14}^{844} - 2X_{14}^{847} + 3X_{14}^{916}$,
$x.X_{14}^{945} = 2X_{14}^{942} + X_{14}^{924} + 2X_{14}^{944}$,
$x.X_{14}^{947} = 3X_{14}^{944} - 2X_{14}^{450} + 2X_{14}^{467} + 2X_{14}^{857} + 3X_{14}^{926} + X_{14}^{946}$,
$x.X_{14}^{949} = 4X_{14}^{946} - 3X_{14}^{164} + 3X_{14}^{768} - 3X_{14}^{778} - 8X_{14}^{445} + 8X_{14}^{479} + 8X_{14}^{862} + 6X_{14}^{928}$,
$x.X_{14}^{969} = 3X_{14}^{966} + 2X_{14}^{882} + 3X_{14}^{942} + 2X_{14}^{968}$,
$x.X_{14}^{971} = 4X_{14}^{968} - 3X_{14}^{176} + 3X_{14}^{788} + 8X_{14}^{887} + 6X_{14}^{944} + X_{14}^{970}$,
$x.X_{14}^{973} = 5X_{14}^{970} + 4X_{14}^{465} - 4X_{14}^{516} - 15X_{14}^{168} + 15X_{14}^{792} + 20X_{14}^{892} + 10X_{14}^{946}$,
$x.X_{14}^{987} = 2X_{14}^{986} + X_{14}^{181}$, $x.X_{14}^{990} = X_{14}^{986} + X_{14}^{989}$,
$x.X_{14}^{996} = 3X_{14}^{995} + 3X_{14}^{226} - 2X_{14}^{540}$, $x.X_{14}^{999} = X_{14}^{995} + 2X_{14}^{998} + X_{14}^{235}$,
$x.X_{14}^{1002} = 2X_{14}^{998} + X_{14}^{986} + X_{14}^{1001}$, $x.X_{14}^{1005} = 3X_{14}^{1001} + X_{14}^{989}$,
$x.X_{14}^{1011} = 2X_{14}^{1010} + X_{14}^{289}$, $x.X_{14}^{1013} = X_{14}^{1010} + X_{14}^{1012}$,
$x.X_{14}^{1015} = 2X_{14}^{1012}$, $x.X_{14}^{1023} = X_{14}^{1020} + 2X_{14}^{1022} + X_{14}^{325}$,
$x.X_{14}^{1025} = 2X_{14}^{1022} + X_{14}^{1010} + X_{14}^{1024}$,
$x.X_{14}^{1027} = 3X_{14}^{1024} + 4X_{14}^{181} - 2X_{14}^{532} + 2X_{14}^{986} - 2X_{14}^{989} + 3X_{14}^{1012}$,
$x.X_{14}^{1041} = 2X_{14}^{1038} + X_{14}^{1020} + 2X_{14}^{1040} + X_{14}^{379}$,
$x.X_{14}^{1043} = 3X_{14}^{1040} + 2X_{14}^{226} - 2X_{14}^{540} + 2X_{14}^{995} + 3X_{14}^{1022} + X_{14}^{1042}$,
$x.X_{14}^{1045} = 4X_{14}^{1042} - 6X_{14}^{7} + 3X_{14}^{816} - 3X_{14}^{832} + 16X_{14}^{235} - 8X_{14}^{535} + 8X_{14}^{998} + 6X_{14}^{1024}$,
$x.X_{14}^{1055} = X_{14}^{1054}$, $x.X_{14}^{1056} = X_{14}^{1054}$,
$x.X_{14}^{1061} = X_{14}^{1059} + X_{14}^{1060}$, $x.X_{14}^{1062} = 2X_{14}^{1060} + X_{14}^{1054}$,
$x.X_{14}^{1071} = 2X_{14}^{1069} + X_{14}^{1059} + X_{14}^{1070}$,
$x.X_{14}^{1072} = 3X_{14}^{1070} - 4X_{14}^{200} + 4X_{14}^{545} - 2X_{14}^{22} - 2X_{14}^{847} + 2X_{14}^{289} + 2X_{14}^{1010} + 3X_{14}^{1060}$,
$x.X_{14}^{1085} = 3X_{14}^{1083} - 2X_{14}^{212} + 2X_{14}^{587} + 2X_{14}^{1020} + 3X_{14}^{1069} + X_{14}^{1084}$,
$x.X_{14}^{1086} = 4X_{14}^{1084} + 3X_{14}^{220} - 3X_{14}^{570} + 3X_{14}^{33} + 3X_{14}^{855} - 8X_{14}^{204} + 8X_{14}^{593} + 8X_{14}^{325}$
$+8X_{14}^{1022} + 6X_{14}^{1070}$,
$x.X_{14}^{1103} = 4X_{14}^{1101} + 3X_{14}^{72} + 3X_{14}^{882} + 8X_{14}^{1038} + 6X_{14}^{1083} + X_{14}^{1102}$,

$x.X_{14}^{1104} = 5X_{14}^{1102} - 4X_{14}^{275} + 4X_{14}^{585} + 15X_{14}^{83} + 15X_{14}^{885} + 20X_{14}^{379} + 20X_{14}^{1040} + 10X_{14}^{1084}$,
$x.X_{14}^{1115} = X_{14}^{1114}$, $x.X_{14}^{1119} = 2X_{14}^{1118} + X_{14}^{1114}$,
$x.X_{14}^{1125} = 3X_{14}^{1124} + 6X_{14}^{36} - 2X_{14}^{648} + 6X_{14}^{653} + 2X_{14}^{1059} + 3X_{14}^{1118}$,
$x.X_{14}^{1133} = 4X_{14}^{1132} - 3X_{14}^{311} + 6X_{14}^{319} + 3X_{14}^{924} + 8X_{14}^{86} + 16X_{14}^{683} + 8X_{14}^{1069} + 6X_{14}^{1124}$,
$x.X_{14}^{1143} = 5X_{14}^{1142} + 4X_{14}^{78} + 4X_{14}^{681} + 15X_{14}^{373} + 15X_{14}^{942} + 20X_{14}^{725} + 20X_{14}^{1083} + 10X_{14}^{1132}$,
$x.X_{14}^{1155} = 6X_{14}^{1154} + 5X_{14}^{369} + 24X_{14}^{723} + 45X_{14}^{966} + 40X_{14}^{1101} + 15X_{14}^{1142}$.

E.3 Degree 8 in the adjoint representation of sl(2)

$x.X_8^2 = X_8^1$, $x.X_8^{13} = X_8^9 + X_8^{12}$,
$x.X_8^{18} = -4X_8^9 + X_8^{17}$, $x.X_8^{23} = -2X_8^{12} + X_8^{17} - 2X_8^{22}$,
$x.X_8^{29} = 2X_8^{22} - X_8^1$, $x.X_8^{34} = 2X_8^{27} + X_8^{12} + X_8^{33}$,
$x.X_8^{35} = 2X_8^{28} + X_8^{17} - 4X_8^{33}$, $x.X_8^{48} = X_8^9 - 2X_8^{39} + X_8^{47}$,
$x.X_8^{54} = -X_8^{33} - 2X_8^{45} + 2X_8^{53} - X_8^{47}$,
$x.X_8^{57} = X_8^{12} + X_8^{47} - 2X_8^{56}$,
$x.X_8^{63} = -X_8^{27} + X_8^{53} - 2X_8^{61} + X_8^{62}$,
$x.X_8^{68} = X_8^{17} - 4X_8^{47} + 2X_8^1 - 2X_8^{67}$,
$x.X_8^{74} = -X_8^{28} - 4X_8^{53} + 2X_8^{22} - 2X_8^{72} + X_8^{73}$,
$x.X_8^{78} = -X_8^{62} + 4X_8^{66} + 2X_8^{56} + X_8^{72} + X_8^{77}$,
$x.X_8^{79} = X_8^{22} - 2X_8^{56} + X_8^{67}$,
$x.X_8^{84} = X_8^{27} - 2X_8^{61} + X_8^{72} + X_8^{83}$,
$x.X_8^{85} = X_8^{28} - 2X_8^{62} + X_8^{73} - 4X_8^{83}$,
$x.X_8^{89} = -X_8^{53} - 2X_8^{66} + X_8^{77} + X_8^{83} - 2X_8^{88}$,
$x.X_8^{96} = X_8^{33} + 2X_8^{83} + X_8^9 - 2X_8^{95}$,
$x.X_8^{101} = -X_8^{45} + 2X_8^{88} + X_8^{39} + X_8^{95}$,
$x.X_8^{106} = -6X_8^{67} - 6X_8^1 + 4X_8^{22}$,
$x.X_8^{111} = -6X_8^{72} + 6X_8^{12} + 4X_8^{27} + X_8^{110}$,
$x.X_8^{112} = -6X_8^{73} + 6X_8^{17} + 4X_8^{28} - 4X_8^{110}$,
$x.X_8^{116} = -6X_8^{77} + 6X_8^{47} - 4X_8^{53} + X_8^{110} - 2X_8^{115}$,
$x.X_8^{125} = -4X_8^{83} + 2X_8^{33} + X_8^{110} - 2X_8^{124}$,
$x.X_8^{130} = -4X_8^{88} - 2X_8^{45} + X_8^{115} + X_8^{124}$,
$x.X_8^{140} = -2X_8^{95} + 2X_8^{124}$, $x.X_8^{169} = X_8^{161} - 2X_8^{167} + X_8^{168}$,
$x.X_8^{175} = -2X_8^{167} + X_8^{172} + X_8^{174}$, $x.X_8^{176} = -2X_8^{168} - 4X_8^{174} + 2X_8^{172}$,
$x.X_8^{185} = -2X_8^{161} - 2X_8^{183} + X_8^{184}$,
$x.X_8^{191} = -2X_8^{167} + X_8^{183} + X_8^{188} + X_8^{190}$,
$x.X_8^{192} = -2X_8^{168} + X_8^{184} - 4X_8^{190} + 2X_8^{188}$,
$x.X_8^{196} = -2X_8^{172} - 2X_8^{188} + X_8^{195}$,
$x.X_8^{198} = -2X_8^{174} - 2X_8^{190} + X_8^{195} - 2X_8^{197}$,
$x.X_8^{207} = X_8^{183} + X_8^{204} + X_8^{206}$, $x.X_8^{208} = X_8^{184} - 4X_8^{206} + 2X_8^{204}$,
$x.X_8^{212} = X_8^{188} + X_8^{204} + X_8^{211}$,
$x.X_8^{214} = X_8^{190} + X_8^{206} + X_8^{211} - 2X_8^{213}$,
$x.X_8^{219} = X_8^{195} - 2X_8^{211} - 2X_8^{218}$, $x.X_8^{221} = X_8^{197} - 2X_8^{213} + X_8^{218}$,
$x.X_8^{233} = X_8^{161} - 2X_8^{231} + X_8^{232}$,
$x.X_8^{239} = X_8^{167} + X_8^{231} + X_8^{236} + X_8^{238}$,

$x.X_8^{240} = X_8^{168} + X_8^{232} - 4X_8^{238} + 2X_8^{236},$
$x.X_8^{244} = X_8^{172} - 2X_8^{236} + X_8^{243},$
$x.X_8^{246} = X_8^{174} - 2X_8^{238} + X_8^{243} - 2X_8^{245},$
$x.X_8^{255} = X_8^{183} - 2X_8^{231} + X_8^{252} + X_8^{254},$
$x.X_8^{256} = X_8^{184} - 2X_8^{232} - 4X_8^{254} + 2X_8^{252},$
$x.X_8^{260} = X_8^{188} - 2X_8^{236} + X_8^{252} + X_8^{259},$
$x.X_8^{262} = X_8^{190} - 2X_8^{238} + X_8^{254} + X_8^{259} - 2X_8^{261},$
$x.X_8^{267} = X_8^{195} - 2X_8^{243} - 2X_8^{259} - 2X_8^{266},$
$x.X_8^{269} = X_8^{197} - 2X_8^{245} - 2X_8^{261} + X_8^{266},$
$x.X_8^{276} = X_8^{204} + X_8^{252} + X_8^{275},$
$x.X_8^{278} = X_8^{206} + X_8^{254} + X_8^{275} - 2X_8^{277},$
$x.X_8^{283} = X_8^{211} + X_8^{259} + X_8^{275} - 2X_8^{282},$
$x.X_8^{285} = X_8^{213} + X_8^{261} + X_8^{277} + X_8^{282},$
$x.X_8^{290} = X_8^{218} + X_8^{266} - 2X_8^{282},$ $x.X_8^{300} = -4X_8^{252} + 2X_8^{204} + X_8^{299},$
$x.X_8^{302} = -4X_8^{254} + 2X_8^{206} + X_8^{299} - 2X_8^{301},$
$x.X_8^{307} = -4X_8^{259} + 2X_8^{211} + X_8^{299} - 2X_8^{306},$
$x.X_8^{309} = -4X_8^{261} + 2X_8^{213} + X_8^{301} + X_8^{306},$
$x.X_8^{314} = -4X_8^{266} + 2X_8^{218} - 2X_8^{306},$ $x.X_8^{323} = -2X_8^{275} + X_8^{299} - 2X_8^{322},$
$x.X_8^{325} = -2X_8^{277} + X_8^{301} + X_8^{322},$ $x.X_8^{330} = -2X_8^{282} + X_8^{306} + X_8^{322},$
$x.X_8^{369} = X_8^{361} - 2X_8^{367} + X_8^{368},$
$x.X_8^{377} = -4X_8^{361} - 2X_8^{161} - 2X_8^{375} + X_8^{376},$
$x.X_8^{383} = -2X_8^{367} + X_8^{375} + X_8^{380} + X_8^{382},$
$x.X_8^{384} = -2X_8^{368} + X_8^{376} - 4X_8^{382} + 2X_8^{380},$
$x.X_8^{388} = 2X_8^{380} + X_8^{172} + X_8^{387},$
$x.X_8^{390} = 2X_8^{382} + X_8^{174} + X_8^{387} - 2X_8^{389},$
$x.X_8^{399} = -6X_8^{375} - 2X_8^{183} + X_8^{396} + X_8^{398},$
$x.X_8^{400} = -6X_8^{376} - 2X_8^{184} - 4X_8^{398} + 2X_8^{396},$
$x.X_8^{404} = -4X_8^{380} + X_8^{396} + X_8^{403},$
$x.X_8^{406} = -4X_8^{382} + X_8^{398} + X_8^{403} - 2X_8^{405},$
$x.X_8^{411} = -2X_8^{387} + 2X_8^{403} + X_8^{195} - 2X_8^{410},$
$x.X_8^{413} = -2X_8^{389} + 2X_8^{405} + X_8^{197} + X_8^{410},$
$x.X_8^{418} = 3X_8^{410} + 3X_8^{218} - 2X_8^{322},$
$x.X_8^{441} = X_8^{361} - 2X_8^{433} - 2X_8^{439} + X_8^{440},$
$x.X_8^{447} = X_8^{367} + X_8^{439} + X_8^{444} + X_8^{446},$
$x.X_8^{448} = X_8^{368} + X_8^{440} - 4X_8^{446} + 2X_8^{444},$
$x.X_8^{455} = X_8^{375} - 4X_8^{439} - 2X_8^{231} + X_8^{452} + X_8^{454},$
$x.X_8^{456} = X_8^{376} - 4X_8^{440} - 2X_8^{232} - 4X_8^{454} + 2X_8^{452},$
$x.X_8^{460} = X_8^{380} - 2X_8^{444} + X_8^{452} + X_8^{459},$
$x.X_8^{462} = X_8^{382} - 2X_8^{446} + X_8^{454} + X_8^{459} - 2X_8^{461},$
$x.X_8^{467} = X_8^{387} + 2X_8^{459} + X_8^{243} - 2X_8^{466},$
$x.X_8^{469} = X_8^{389} + 2X_8^{461} + X_8^{245} + X_8^{466},$
$x.X_8^{476} = X_8^{396} - 6X_8^{452} + 4X_8^{188} - 6X_8^{252} + X_8^{475},$
$x.X_8^{478} = X_8^{398} - 6X_8^{454} + 4X_8^{190} - 6X_8^{254} + X_8^{475} - 2X_8^{477},$
$x.X_8^{483} = X_8^{403} - 4X_8^{459} + 2X_8^{211} - 2X_8^{275} + X_8^{475} - 2X_8^{482},$
$x.X_8^{485} = X_8^{405} - 4X_8^{461} + 2X_8^{213} - 2X_8^{277} + X_8^{477} + X_8^{482},$

$x.X_8^{490} = X_8^{410} - 2X_8^{466} + 2X_8^{482} + X_8^{266},$
$x.X_8^{507} = -8X_8^{475} + 16X_8^{195} - 12X_8^{299} + 6X_8^{403} - 2X_8^{506},$
$x.X_8^{509} = -8X_8^{477} + 16X_8^{197} - 12X_8^{301} + 6X_8^{405} + X_8^{506},$
$x.X_8^{514} = -6X_8^{482} + 6X_8^{218} - 6X_8^{322} + 4X_8^{410} + X_8^{506},$
$x.X_8^{542} = X_8^{540} - 2X_8^{541},$ $x.X_8^{546} = -2X_8^{540} - 2X_8^{545},$
$x.X_8^{547} = -2X_8^{541} + 2X_8^{545} + X_8^{1},$ $x.X_8^{564} = -2X_8^{558} - 4X_8^{563} - 2X_8^{9},$
$x.X_8^{569} = X_8^{563} + X_8^{567} - 2X_8^{568},$ $x.X_8^{575} = -4X_8^{563} + X_8^{573} - 2X_8^{574},$
$x.X_8^{579} = -2X_8^{567} + X_8^{573} - 2X_8^{578},$
$x.X_8^{580} = -2X_8^{568} + X_8^{574} + 2X_8^{578} + X_8^{22},$
$x.X_8^{584} = 2X_8^{578} - X_8^{545} + X_8^{583},$ $x.X_8^{594} = X_8^{558} - 4X_8^{593} - 2X_8^{39},$
$x.X_8^{599} = X_8^{563} - 2X_8^{593} + X_8^{597} - 2X_8^{598},$
$x.X_8^{603} = X_8^{567} + X_8^{597} - 2X_8^{602},$
$x.X_8^{604} = X_8^{568} + X_8^{598} + 2X_8^{602} + X_8^{56},$
$x.X_8^{609} = X_8^{573} - 4X_8^{597} - 2X_8^{540} - 2X_8^{608},$
$x.X_8^{610} = X_8^{574} - 4X_8^{598} - 2X_8^{541} + 2X_8^{608} + X_8^{67},$
$x.X_8^{614} = X_8^{578} - 2X_8^{602} + X_8^{608} + X_8^{613},$
$x.X_8^{619} = X_8^{583} + 2X_8^{613} - 2X_8^{1} + X_8^{540} - X_8^{545},$
$x.X_8^{626} = -6X_8^{608} - 6X_8^{545} + 4X_8^{578} + X_8^{625},$
$x.X_8^{631} = -4X_8^{613} + 2X_8^{583} + X_8^{625},$ $x.X_8^{660} = 2X_8^{657} + X_8^{558} - 2X_8^{659},$
$x.X_8^{666} = -4X_8^{657} - 2X_8^{172} + 2X_8^{218} - 2X_8^{567} + X_8^{663} - 2X_8^{665},$
$x.X_8^{668} = -2X_8^{659} + 2X_8^{665} + X_8^{9} + X_8^{563} + X_8^{667},$
$x.X_8^{670} = 3X_8^{667} + 2X_8^{161} - 2X_8^{322} + 3X_8^{12} + 3X_8^{567},$
$x.X_8^{675} = -8X_8^{663} - 6X_8^{573} - 2X_8^{674},$
$x.X_8^{677} = -6X_8^{665} - 2X_8^{188} + 2X_8^{211} - 2X_8^{578} + X_8^{674} + X_8^{676},$
$x.X_8^{679} = -4X_8^{667} + 2X_8^{676} + X_8^{17} + X_8^{573},$
$x.X_8^{702} = X_8^{657} - 2X_8^{696} + X_8^{699} - 2X_8^{701},$
$x.X_8^{704} = X_8^{659} + 2X_8^{701} + X_8^{39} + X_8^{593} + X_8^{703},$
$x.X_8^{708} = X_8^{663} - 6X_8^{699} + 4X_8^{197} - 4X_8^{243} - 6X_8^{597} - 2X_8^{707},$
$x.X_8^{710} = X_8^{665} - 4X_8^{701} + 2X_8^{213} - 2X_8^{236} - 2X_8^{602} + X_8^{707} + X_8^{709},$
$x.X_8^{712} = X_8^{667} - 2X_8^{703} + 2X_8^{709} + X_8^{47} + X_8^{597},$
$x.X_8^{721} = X_8^{676} - 6X_8^{709} + 4X_8^{183} - 4X_8^{275} + 4X_8^{22} + 4X_8^{578} + 6X_8^{206} - 6X_8^{252} - 6X_8^{613}$
$+X_8^{718},$
$x.X_8^{733} = -10X_8^{718} + 40X_8^{184} - 40X_8^{299} + 30X_8^{17} + 30X_8^{573} - 20X_8^{625} + 8X_8^{676},$
$x.X_8^{761} = -2X_8^{757} + 3X_8^{760} + 2X_8^{361} + 3X_8^{657},$
$x.X_8^{765} = -6X_8^{760} + 8X_8^{27} - 4X_8^{387} + 4X_8^{410} + 6X_8^{33} - 12X_8^{380} - 6X_8^{667} + 2X_8^{764}$
$+X_8^{663},$
$x.X_8^{770} = -10X_8^{764} + 10X_8^{28} - 10X_8^{403} - 12X_8^{396} - 12X_8^{676} + X_8^{769},$
$x.X_8^{785} = X_8^{757} + 3X_8^{784} + 2X_8^{433} + 3X_8^{696},$
$x.X_8^{788} = X_8^{760} - 4X_8^{784} - 2X_8^{45} - 4X_8^{444} - 2X_8^{703} + 2X_8^{787} + X_8^{699},$
$x.X_8^{792} = X_8^{764} - 8X_8^{787} - 6X_8^{62} + 6X_8^{405} - 16X_8^{53} - 16X_8^{459} - 12X_8^{452} - 12X_8^{709}$
$+X_8^{791},$
$x.X_8^{797} = X_8^{769} - 12X_8^{791} - 90X_8^{73} + 48X_8^{398} - 80X_8^{475} + 10X_8^{674} - 30X_8^{718}.$

E.4 Degree 9 in the natural representation of sl(3)

$x_1 \cdot X_9^{18} = X_9^{15}$, $x_1 \cdot X_9^{35} = 2X_9^{32} - X_9^{27}$,
$x_1 \cdot X_9^{45} = X_9^{27}$, $x_1 \cdot X_9^{50} = X_9^{32} + X_9^{48}$,
$x_1 \cdot X_9^{69} = X_9^{15}$, $x_1 \cdot X_9^{75} = X_9^{21}$,
$x_1 \cdot X_9^{81} = X_9^{27}$, $x_1 \cdot X_9^{86} = X_9^{32} + X_9^{84}$,
$x_1 \cdot X_9^{102} = X_9^{48} + X_9^{84}$, $x_1 \cdot X_9^{111} = 0$,
$x_1 \cdot X_9^{117} = 0$, $x_1 \cdot X_9^{122} = X_9^{120}$,
$x_1 \cdot X_9^{128} = X_9^{127}$, $x_1 \cdot X_9^{134} = X_9^{132}$,
$x_1 \cdot X_9^{139} = 2X_9^{137} + X_9^{127}$, $x_1 \cdot X_9^{145} = X_9^{127}$,
$x_1 \cdot X_9^{150} = X_9^{132}$, $x_1 \cdot X_9^{155} = X_9^{137} + X_9^{154}$.
$x_1 \cdot X_9^{179} = 2X_9^{176} - X_9^{171}$, $x_1 \cdot X_9^{196} = 3X_9^{193} + 3X_9^{182} - 2X_9^{188}$,
$x_1 \cdot X_9^{200} = X_9^{182} + X_9^{199}$, $x_1 \cdot X_9^{206} = X_9^{188} + X_9^{204}$,
$x_1 \cdot X_9^{211} = X_9^{193} + 2X_9^{209} + X_9^{199}$, $x_1 \cdot X_9^{219} = X_9^{165}$,
$x_1 \cdot X_9^{225} = X_9^{171}$, $x_1 \cdot X_9^{230} = X_9^{176} + X_9^{228}$,
$x_1 \cdot X_9^{236} = X_9^{182} + X_9^{235}$, $x_1 \cdot X_9^{242} = X_9^{188} + X_9^{240}$,
$x_1 \cdot X_9^{247} = X_9^{193} + 2X_9^{245} + X_9^{235}$, $x_1 \cdot X_9^{253} = X_9^{199} + X_9^{235}$,
$x_1 \cdot X_9^{258} = X_9^{204} + X_9^{240}$, $x_1 \cdot X_9^{263} = X_9^{209} + X_9^{245} + X_9^{262}$,
$x_1 \cdot X_9^{272} = X_9^{271}$, $x_1 \cdot X_9^{278} = X_9^{276}$,
$x_1 \cdot X_9^{283} = 2X_9^{281} + X_9^{271}$, $x_1 \cdot X_9^{295} = 2X_9^{293}$,
$x_1 \cdot X_9^{311} = X_9^{293} + X_9^{310}$, $x_1 \cdot X_9^{336} = 2X_9^{228} - X_9^{120}$,
$x_1 \cdot X_9^{343} = 2X_9^{235} - X_9^{127}$, $x_1 \cdot X_9^{348} = 2X_9^{240} - X_9^{132}$,
$x_1 \cdot X_9^{353} = 2X_9^{245} - X_9^{137} + X_9^{352}$, $x_1 \cdot X_9^{370} = 2X_9^{262} - X_9^{154} + X_9^{352}$,
$x_1 \cdot X_9^{379} = X_9^{271}$, $x_1 \cdot X_9^{384} = X_9^{276}$,
$x_1 \cdot X_9^{389} = X_9^{281} + X_9^{388}$, $x_1 \cdot X_9^{401} = X_9^{293} + X_9^{400}$,
$x_1 \cdot X_9^{418} = X_9^{310} + X_9^{400}$, $x_1 \cdot X_9^{468} = X_9^{450} + X_9^{465}$,
$x_1 \cdot X_9^{485} = 2X_9^{482} - X_9^{477}$, $x_1 \cdot X_9^{501} = 2X_9^{465} + X_9^{15}$,
$x_1 \cdot X_9^{507} = X_9^{471}$, $x_1 \cdot X_9^{513} = X_9^{477}$,
$x_1 \cdot X_9^{518} = X_9^{482} + X_9^{516}$, $x_1 \cdot X_9^{524} = X_9^{523}$,
$x_1 \cdot X_9^{530} = X_9^{528}$, $x_1 \cdot X_9^{535} = 2X_9^{533} + X_9^{523}$,
$x_1 \cdot X_9^{570} = 2X_9^{516}$, $x_1 \cdot X_9^{577} = X_9^{523}$,
$x_1 \cdot X_9^{582} = X_9^{528}$, $x_1 \cdot X_9^{587} = X_9^{533} + X_9^{586}$,
$x_1 \cdot X_9^{599} = X_9^{598}$, $x_1 \cdot X_9^{647} = X_9^{629} + 2X_9^{644} - X_9^{639}$,
$x_1 \cdot X_9^{664} = 3X_9^{661} + 3X_9^{650} - 2X_9^{656}$, $x_1 \cdot X_9^{669} = 2X_9^{633} + X_9^{165}$,
$x_1 \cdot X_9^{675} = 2X_9^{639} + X_9^{171}$, $x_1 \cdot X_9^{680} = 2X_9^{644} + X_9^{176} + X_9^{678}$,
$x_1 \cdot X_9^{686} = X_9^{650} + X_9^{685}$, $x_1 \cdot X_9^{692} = X_9^{656} + X_9^{690}$,
$x_1 \cdot X_9^{697} = X_9^{661} + 2X_9^{695} + X_9^{685}$, $x_1 \cdot X_9^{709} = 2X_9^{707}$,
$x_1 \cdot X_9^{732} = 3X_9^{678} - 2X_9^{84} + 3X_9^{228}$, $x_1 \cdot X_9^{739} = 2X_9^{685} - X_9^{127} + X_9^{271}$,
$x_1 \cdot X_9^{744} = 2X_9^{690} - X_9^{132} + X_9^{276}$,
$x_1 \cdot X_9^{749} = 2X_9^{695} - X_9^{137} + X_9^{281} + X_9^{748}$,
$x_1 \cdot X_9^{761} = X_9^{707} + X_9^{760}$,
$x_1 \cdot X_9^{820} = 3X_9^{748} - 3X_9^{154} + 3X_9^{388} - 2X_9^{586}$,
$x_1 \cdot X_9^{832} = 2X_9^{760} - X_9^{598}$, $x_1 \cdot X_9^{880} = X_9^{878}$,
$x_1 \cdot X_9^{885} = X_9^{878}$, $x_1 \cdot X_9^{902} = X_9^{895} + X_9^{900}$,

$x_1 \cdot X_9^{907} = 2X_9^{905} - X_9^{903},$
$x_1 \cdot X_9^{916} = X_9^{895} + X_9^{914},$
$x_1 \cdot X_9^{921} = X_9^{900} + X_9^{914},$
$x_1 \cdot X_9^{924} = X_9^{903},$
$x_1 \cdot X_9^{926} = X_9^{905} + X_9^{925},$
$x_1 \cdot X_9^{933} = 2X_9^{931} + X_9^{905} - X_9^{925},$
$x_1 \cdot X_9^{941} = 2X_9^{939} - X_9^{937},$
$x_1 \cdot X_9^{944} = X_9^{937},$
$x_1 \cdot X_9^{946} = X_9^{939} + X_9^{945},$
$x_1 \cdot X_9^{970} = 2X_9^{914} - X_9^{878},$
$x_1 \cdot X_9^{981} = 2X_9^{925},$
$x_1 \cdot X_9^{987} = 2X_9^{931} + X_9^{905} + X_9^{986},$
$x_1 \cdot X_9^{1000} = X_9^{937},$
$x_1 \cdot X_9^{1002} = X_9^{939} + X_9^{1001},$
$x_1 \cdot X_9^{1008} = X_9^{945} + X_9^{1001},$
$x_1 \cdot X_9^{1040} = X_9^{1038},$
$x_1 \cdot X_9^{1047} = 2X_9^{1045} - X_9^{1043},$
$x_1 \cdot X_9^{1051} = X_9^{1043},$
$x_1 \cdot X_9^{1053} = X_9^{1045} + X_9^{1052},$
$x_1 \cdot X_9^{1062} = X_9^{1038},$
$x_1 \cdot X_9^{1067} = X_9^{1043},$
$x_1 \cdot X_9^{1069} = X_9^{1045} + X_9^{1068},$
$x_1 \cdot X_9^{1076} = X_9^{1052} + X_9^{1068},$
$x_1 \cdot X_9^{1082} = X_9^{1081},$
$x_1 \cdot X_9^{1089} = X_9^{1081},$
$x_1 \cdot X_9^{1120} = X_9^{1112} + X_9^{1118},$
$x_1 \cdot X_9^{1136} = X_9^{1112} + X_9^{1134},$
$x_1 \cdot X_9^{1142} = X_9^{1118} + X_9^{1134},$
$x_1 \cdot X_9^{1152} = X_9^{1150},$
$x_1 \cdot X_9^{1159} = 2X_9^{1157} - X_9^{1155},$
$x_1 \cdot X_9^{1163} = X_9^{1155},$
$x_1 \cdot X_9^{1165} = X_9^{1157} + X_9^{1164},$
$x_1 \cdot X_9^{1182} = 2X_9^{1134},$
$x_1 \cdot X_9^{1198} = X_9^{1150},$
$x_1 \cdot X_9^{1203} = X_9^{1155},$
$x_1 \cdot X_9^{1205} = X_9^{1157} + X_9^{1204},$
$x_1 \cdot X_9^{1212} = X_9^{1164} + X_9^{1204},$
$x_1 \cdot X_9^{1219} = 0,$
$x_1 \cdot X_9^{1221} = X_9^{1220},$
$x_1 \cdot X_9^{1226} = X_9^{1225},$
$x_1 \cdot X_9^{1233} = X_9^{1225},$
$x_1 \cdot X_9^{1263} = X_9^{1255} + 2X_9^{1261} - X_9^{1259},$
$x_1 \cdot X_9^{1272} = X_9^{1248} + X_9^{1270},$
$x_1 \cdot X_9^{1279} = X_9^{1255} + 2X_9^{1277} - X_9^{1275},$
$x_1 \cdot X_9^{1283} = X_9^{1259} + X_9^{1275},$
$x_1 \cdot X_9^{1285} = X_9^{1261} + X_9^{1277} + X_9^{1284},$
$x_1 \cdot X_9^{1295} = 2X_9^{1293} - X_9^{1291},$
$x_1 \cdot X_9^{1306} = X_9^{1298} + X_9^{1305},$
$x_1 \cdot X_9^{1318} = 2X_9^{1270} + X_9^{1038},$
$x_1 \cdot X_9^{1323} = 2X_9^{1275} + X_9^{1043},$
$x_1 \cdot X_9^{1325} = 2X_9^{1277} + X_9^{1045} + X_9^{1324},$
$x_1 \cdot X_9^{1332} = 2X_9^{1284} + X_9^{1052} + X_9^{1324},$
$x_1 \cdot X_9^{1339} = X_9^{1291},$
$x_1 \cdot X_9^{1341} = X_9^{1293} + X_9^{1340},$
$x_1 \cdot X_9^{1346} = X_9^{1298} + X_9^{1345},$
$x_1 \cdot X_9^{1353} = X_9^{1305} + X_9^{1345},$
$x_1 \cdot X_9^{1362} = X_9^{1361},$
$x_1 \cdot X_9^{1396} = 3X_9^{1324} + 3X_9^{1068} - 2X_9^{1204},$
$x_1 \cdot X_9^{1412} = 2X_9^{1340} - X_9^{1220},$
$x_1 \cdot X_9^{1417} = 2X_9^{1345} - X_9^{1225},$
$x_1 \cdot X_9^{1433} = X_9^{1361},$
$x_1 \cdot X_9^{1496} = X_9^{1480} + X_9^{1494},$
$x_1 \cdot X_9^{1503} = 2X_9^{1501} - X_9^{1499},$
$x_1 \cdot X_9^{1518} = 2X_9^{1494} - X_9^{878} + X_9^{1150},$
$x_1 \cdot X_9^{1523} = X_9^{1499},$
$x_1 \cdot X_9^{1525} = X_9^{1501} + X_9^{1524},$
$x_1 \cdot X_9^{1530} = X_9^{1529},$
$x_1 \cdot X_9^{1556} = 2X_9^{1524},$
$x_1 \cdot X_9^{1561} = X_9^{1529},$
$x_1 \cdot X_9^{1608} = 2X_9^{1592} + X_9^{1248} + X_9^{1606},$
$x_1 \cdot X_9^{1615} = X_9^{1599} + 2X_9^{1613} - X_9^{1611},$
$x_1 \cdot X_9^{1630} = 3X_9^{1606} + 2X_9^{900} + 3X_9^{1270},$
$x_1 \cdot X_9^{1635} = 2X_9^{1611} + X_9^{903} + X_9^{1291},$
$x_1 \cdot X_9^{1637} = 2X_9^{1613} + X_9^{905} + X_9^{1293} + X_9^{1636},$
$x_1 \cdot X_9^{1642} = X_9^{1618} + X_9^{1641},$
$x_1 \cdot X_9^{1668} = 3X_9^{1636} + 2X_9^{945} - 2X_9^{1204} + 3X_9^{925} + 3X_9^{1340},$
$x_1 \cdot X_9^{1673} = 2X_9^{1641} - 2X_9^{878} + 2X_9^{937} - X_9^{1225} + X_9^{1361},$
$x_1 \cdot X_9^{1745} = 2X_9^{1744} + X_9^{21},$
$x_1 \cdot X_9^{1754} = X_9^{1744} + X_9^{1753},$
$x_1 \cdot X_9^{1761} = X_9^{1760},$
$x_1 \cdot X_9^{1772} = 2X_9^{1771} + X_9^{165},$
$x_1 \cdot X_9^{1781} = X_9^{1771} + X_9^{1780},$
$x_1 \cdot X_9^{1789} = X_9^{1788},$
$x_1 \cdot X_9^{1800} = 2X_9^{1780} - X_9^{1760},$
$x_1 \cdot X_9^{1808} = X_9^{1788},$

$x_1 \cdot X_9^{1834} = 2X_9^{1833} + X_9^{471},$ $x_1 \cdot X_9^{1845} = X_9^{1833} + X_9^{1844},$
$x_1 \cdot X_9^{1849} = X_9^{1848},$ $x_1 \cdot X_9^{1862} = 2X_9^{1844},$
$x_1 \cdot X_9^{1866} = X_9^{1848},$ $x_1 \cdot X_9^{1871} = 0,$
$x_1 \cdot X_9^{1888} = X_9^{1882} + 2X_9^{1887} + X_9^{633},$ $x_1 \cdot X_9^{1899} = 2X_9^{1887} + X_9^{1771} + X_9^{1898},$
$x_1 \cdot X_9^{1903} = X_9^{1891} + X_9^{1902},$ $x_1 \cdot X_9^{1916} = 3X_9^{1898} - 2X_9^{1753} + 3X_9^{1780},$
$x_1 \cdot X_9^{1920} = 2X_9^{1902} - X_9^{127} + X_9^{1043} - X_9^{1760} + X_9^{1788},$
$x_1 \cdot X_9^{1925} = X_9^{1907},$ $x_1 \cdot X_9^{1987} = X_9^{1978} + X_9^{1986},$
$x_1 \cdot X_9^{1998} = 2X_9^{1986} - 2X_9^{132} + 2X_9^{1155} + X_9^{27} - X_9^{1225} + X_9^{1848},$
$x_1 \cdot X_9^{2000} = X_9^{1988},$
$x_1 \cdot X_9^{2047} = 2X_9^{2038} - X_9^{154} + X_9^{1259} + X_9^{1891} + X_9^{2046},$
$x_1 \cdot X_9^{2058} = 3X_9^{2046} + 2X_9^{188} - 2X_9^{1204} - 2X_9^{84} + 2X_9^{1298} - 3X_9^{137} + 3X_9^{1275} + 3X_9^{1902},$
$x_1 \cdot X_9^{2060} = 2X_9^{2048} + X_9^{171} - X_9^{1220} - 2X_9^{120} + 2X_9^{1291} + X_9^{1907},$
$x_1 \cdot X_9^{2123} = 2X_9^{2118} + 3X_9^{477} - X_9^{598} + 3X_9^{1499} + X_9^{1988},$
$x_1 \cdot X_9^{2158} = 3X_9^{2153} - 2X_9^{586} + 4X_9^{656} + 2X_9^{1618} + 3X_9^{639} + 6X_9^{1611} + 3X_9^{2048},$
$x_2 \cdot X_9^{18} = 2X_9^{17},$ $x_2 \cdot X_9^{35} = X_9^{17} + X_9^{34},$
$x_2 \cdot X_9^{39} = X_9^{38},$ $x_2 \cdot X_9^{45} = 2X_9^{44} - X_9^{38},$
$x_2 \cdot X_9^{50} = X_9^{44} + X_9^{49},$ $x_2 \cdot X_9^{69} = X_9^{63} + 2X_9^{68} - X_9^{57},$
$x_2 \cdot X_9^{75} = X_9^{57} + X_9^{74},$ $x_2 \cdot X_9^{81} = X_9^{63} + 2X_9^{80} - X_9^{74},$
$x_2 \cdot X_9^{86} = X_9^{68} + X_9^{80} + X_9^{85},$ $x_2 \cdot X_9^{102} = X_9^{96} + X_9^{101},$
$x_2 \cdot X_9^{111} = X_9^{57} + X_9^{110},$ $x_2 \cdot X_9^{117} = X_9^{63} + 2X_9^{116} - X_9^{110},$
$x_2 \cdot X_9^{122} = X_9^{68} + X_9^{116} + X_9^{121},$ $x_2 \cdot X_9^{128} = X_9^{74} + X_9^{110},$
$x_2 \cdot X_9^{134} = X_9^{80} + X_9^{116} + X_9^{133},$ $x_2 \cdot X_9^{139} = X_9^{85} + X_9^{121} + X_9^{133},$
$x_2 \cdot X_9^{145} = X_9^{91},$ $x_2 \cdot X_9^{150} = X_9^{96} + X_9^{149},$
$x_2 \cdot X_9^{155} = X_9^{101} + X_9^{149},$ $x_2 \cdot X_9^{179} = X_9^{17} + X_9^{178},$
$x_2 \cdot X_9^{196} = X_9^{34} + X_9^{178},$ $x_2 \cdot X_9^{200} = X_9^{38},$
$x_2 \cdot X_9^{206} = X_9^{44} + X_9^{205},$ $x_2 \cdot X_9^{211} = X_9^{49} + X_9^{205},$
$x_2 \cdot X_9^{219} = X_9^{57} + X_9^{218},$ $x_2 \cdot X_9^{225} = X_9^{63} + 2X_9^{224} - X_9^{218},$
$x_2 \cdot X_9^{230} = X_9^{68} + X_9^{224} + X_9^{229},$ $x_2 \cdot X_9^{236} = X_9^{74} + X_9^{218},$
$x_2 \cdot X_9^{242} = X_9^{80} + X_9^{224} + X_9^{241},$ $x_2 \cdot X_9^{247} = X_9^{85} + X_9^{229} + X_9^{241},$
$x_2 \cdot X_9^{253} = X_9^{91},$ $x_2 \cdot X_9^{258} = X_9^{96} + X_9^{257},$
$x_2 \cdot X_9^{263} = X_9^{101} + X_9^{257},$ $x_2 \cdot X_9^{272} = X_9^{110} + X_9^{218},$
$x_2 \cdot X_9^{278} = X_9^{116} + X_9^{224} + X_9^{277},$ $x_2 \cdot X_9^{283} = X_9^{121} + X_9^{229} + X_9^{277},$
$x_2 \cdot X_9^{295} = X_9^{133} + X_9^{241} + X_9^{277},$ $x_2 \cdot X_9^{311} = X_9^{149} + X_9^{257},$
$x_2 \cdot X_9^{336} = X_9^{330} + X_9^{335},$ $x_2 \cdot X_9^{343} = X_9^{325},$
$x_2 \cdot X_9^{348} = X_9^{330} + X_9^{347},$ $x_2 \cdot X_9^{353} = X_9^{335} + X_9^{347},$
$x_2 \cdot X_9^{370} = X_9^{364},$ $x_2 \cdot X_9^{379} = X_9^{325},$
$x_2 \cdot X_9^{384} = X_9^{330} + X_9^{383},$ $x_2 \cdot X_9^{389} = X_9^{335} + X_9^{383},$
$x_2 \cdot X_9^{401} = X_9^{347} + X_9^{383},$ $x_2 \cdot X_9^{418} = X_9^{364},$
$x_2 \cdot X_9^{468} = 2X_9^{467},$ $x_2 \cdot X_9^{485} = X_9^{467} + X_9^{484},$
$x_2 \cdot X_9^{501} = X_9^{495} + 2X_9^{500} - X_9^{489},$ $x_2 \cdot X_9^{507} = X_9^{489} + X_9^{506},$
$x_2 \cdot X_9^{513} = X_9^{495} + 2X_9^{512} - X_9^{506},$ $x_2 \cdot X_9^{518} = X_9^{500} + X_9^{512} + X_9^{517},$
$x_2 \cdot X_9^{524} = 2X_9^{506} + X_9^{38},$ $x_2 \cdot X_9^{530} = 2X_9^{512} + X_9^{44} + X_9^{529},$
$x_2 \cdot X_9^{535} = 2X_9^{517} + X_9^{49} + X_9^{529},$ $x_2 \cdot X_9^{570} = X_9^{552} + X_9^{564} + X_9^{569},$
$x_2 \cdot X_9^{577} = 2X_9^{559} + X_9^{91},$ $x_2 \cdot X_9^{582} = 2X_9^{564} + X_9^{96} + X_9^{581},$
$x_2 \cdot X_9^{587} = 2X_9^{569} + X_9^{101} + X_9^{581},$ $x_2 \cdot X_9^{599} = 3X_9^{581} + 3X_9^{149} - 2X_9^{383},$

$x_2 \cdot X_9^{647} = X_9^{467} + X_9^{646},$
$x_2 \cdot X_9^{664} = X_9^{484} + X_9^{646},$
$x_2 \cdot X_9^{669} = X_9^{489} + X_9^{668},$
$x_2 \cdot X_9^{675} = X_9^{495} + 2X_9^{674} - X_9^{668},$
$x_2 \cdot X_9^{680} = X_9^{500} + X_9^{674} + X_9^{679},$
$x_2 \cdot X_9^{686} = X_9^{506} + X_9^{668},$
$x_2 \cdot X_9^{692} = X_9^{512} + X_9^{674} + X_9^{691},$
$x_2 \cdot X_9^{697} = X_9^{517} + X_9^{679} + X_9^{691},$
$x_2 \cdot X_9^{709} = X_9^{529} + 2X_9^{691} + X_9^{205},$
$x_2 \cdot X_9^{732} = X_9^{552} + X_9^{726} + X_9^{731},$
$x_2 \cdot X_9^{739} = X_9^{559} + X_9^{721},$
$x_2 \cdot X_9^{744} = X_9^{564} + X_9^{726} + X_9^{743},$
$x_2 \cdot X_9^{749} = X_9^{569} + X_9^{731} + X_9^{743},$
$x_2 \cdot X_9^{761} = X_9^{581} + 2X_9^{743} + X_9^{257},$
$x_2 \cdot X_9^{820} = X_9^{802} + X_9^{814},$
$x_2 \cdot X_9^{832} = 2X_9^{814} + X_9^{364},$
$x_2 \cdot X_9^{880} = X_9^{872} + X_9^{879},$
$x_2 \cdot X_9^{885} = X_9^{882} + X_9^{884},$
$x_2 \cdot X_9^{902} = -X_9^{872} + X_9^{901},$
$x_2 \cdot X_9^{907} = -X_9^{879} + X_9^{901},$
$x_2 \cdot X_9^{916} = X_9^{872} + X_9^{915},$
$x_2 \cdot X_9^{921} = X_9^{918} + X_9^{920},$
$x_2 \cdot X_9^{924} = -X_9^{882} + 2X_9^{918} - X_9^{884},$
$x_2 \cdot X_9^{926} = -2X_9^{884} + X_9^{920} + X_9^{918},$
$x_2 \cdot X_9^{933} = -X_9^{901} + X_9^{915},$
$x_2 \cdot X_9^{941} = X_9^{879} + X_9^{915},$
$x_2 \cdot X_9^{944} = X_9^{882} + X_9^{918} + X_9^{943},$
$x_2 \cdot X_9^{946} = X_9^{884} + X_9^{920} + X_9^{943},$
$x_2 \cdot X_9^{970} = X_9^{967} + X_9^{969},$
$x_2 \cdot X_9^{981} = X_9^{975} - X_9^{879} + X_9^{967},$
$x_2 \cdot X_9^{987} = X_9^{969} - X_9^{901} + X_9^{975},$
$x_2 \cdot X_9^{1000} = X_9^{967} + X_9^{999},$
$x_2 \cdot X_9^{1002} = X_9^{969} + X_9^{999},$
$x_2 \cdot X_9^{1008} = X_9^{975} - X_9^{872} + X_9^{999},$
$x_2 \cdot X_9^{1040} = X_9^{1039},$
$x_2 \cdot X_9^{1047} = X_9^{1039},$
$x_2 \cdot X_9^{1051} = X_9^{1050},$
$x_2 \cdot X_9^{1053} = X_9^{1050},$
$x_2 \cdot X_9^{1062} = X_9^{1059} + X_9^{1061},$
$x_2 \cdot X_9^{1067} = X_9^{1059} + X_9^{1066},$
$x_2 \cdot X_9^{1069} = X_9^{1061} + X_9^{1066},$
$x_2 \cdot X_9^{1076} = X_9^{1073},$
$x_2 \cdot X_9^{1082} = 2X_9^{1066} - X_9^{1050},$
$x_2 \cdot X_9^{1089} = X_9^{1073},$
$x_2 \cdot X_9^{1120} = X_9^{1119},$
$x_2 \cdot X_9^{1136} = X_9^{1128} + X_9^{1135},$
$x_2 \cdot X_9^{1142} = X_9^{1139} + X_9^{1141},$
$x_2 \cdot X_9^{1152} = X_9^{1128} + X_9^{1151},$
$x_2 \cdot X_9^{1159} = X_9^{1135} + X_9^{1151},$
$x_2 \cdot X_9^{1163} = X_9^{1139} + X_9^{1162},$
$x_2 \cdot X_9^{1165} = X_9^{1141} + X_9^{1162},$
$x_2 \cdot X_9^{1182} = X_9^{1174} + X_9^{1179} + X_9^{1181},$
$x_2 \cdot X_9^{1198} = X_9^{1174} + X_9^{1195} + X_9^{1197},$
$x_2 \cdot X_9^{1203} = X_9^{1179} + X_9^{1195} + X_9^{1202},$
$x_2 \cdot X_9^{1205} = X_9^{1181} + X_9^{1197} + X_9^{1202},$
$x_2 \cdot X_9^{1212} = X_9^{1188} + X_9^{1209},$
$x_2 \cdot X_9^{1219} = 2X_9^{1195} - X_9^{1059} + X_9^{1218},$
$x_2 \cdot X_9^{1221} = 2X_9^{1197} - X_9^{1061} + X_9^{1218},$
$x_2 \cdot X_9^{1226} = 2X_9^{1202} - X_9^{1066} + X_9^{1218},$
$x_2 \cdot X_9^{1233} = 2X_9^{1209} - X_9^{1073},$
$x_2 \cdot X_9^{1263} = X_9^{1119},$
$x_2 \cdot X_9^{1272} = X_9^{1128} + X_9^{1271},$
$x_2 \cdot X_9^{1279} = X_9^{1135} + X_9^{1271},$
$x_2 \cdot X_9^{1283} = X_9^{1139} + X_9^{1282},$
$x_2 \cdot X_9^{1285} = X_9^{1141} + X_9^{1282},$
$x_2 \cdot X_9^{1295} = X_9^{1151} + X_9^{1271},$
$x_2 \cdot X_9^{1306} = X_9^{1162} + X_9^{1282},$
$x_2 \cdot X_9^{1318} = X_9^{1174} + X_9^{1315} + X_9^{1317},$
$x_2 \cdot X_9^{1323} = X_9^{1179} + X_9^{1315} + X_9^{1322},$
$x_2 \cdot X_9^{1325} = X_9^{1181} + X_9^{1317} + X_9^{1322},$
$x_2 \cdot X_9^{1332} = X_9^{1188} + X_9^{1329},$
$x_2 \cdot X_9^{1339} = X_9^{1195} + X_9^{1315} + X_9^{1338},$
$x_2 \cdot X_9^{1341} = X_9^{1197} + X_9^{1317} + X_9^{1338},$
$x_2 \cdot X_9^{1346} = X_9^{1202} + X_9^{1322} + X_9^{1338},$
$x_2 \cdot X_9^{1353} = X_9^{1209} + X_9^{1329},$
$x_2 \cdot X_9^{1362} = X_9^{1218} + 2X_9^{1338} + X_9^{1050} - X_9^{1066},$
$x_2 \cdot X_9^{1396} = X_9^{1388} + X_9^{1393},$
$x_2 \cdot X_9^{1412} = X_9^{1388} + X_9^{1409},$
$x_2 \cdot X_9^{1417} = X_9^{1393} + X_9^{1409},$
$x_2 \cdot X_9^{1433} = 2X_9^{1409},$
$x_2 \cdot X_9^{1496} = X_9^{1488} + X_9^{1495},$
$x_2 \cdot X_9^{1503} = 2X_9^{1495} + X_9^{1119},$
$x_2 \cdot X_9^{1518} = X_9^{1510} + X_9^{1515} + X_9^{1517},$
$x_2 \cdot X_9^{1523} = 2X_9^{1515} + X_9^{1139} + X_9^{1522},$
$x_2 \cdot X_9^{1525} = 2X_9^{1517} + X_9^{1141} + X_9^{1522},$
$x_2 \cdot X_9^{1530} = 3X_9^{1522} - 2X_9^{999} + 3X_9^{1162},$
$x_2 \cdot X_9^{1556} = 2X_9^{1548} + X_9^{1188} + X_9^{1553},$

$x_2 \cdot X_9^{1561} = 3X_9^{1553} + 2X_9^{872} - 2X_9^{999} + 3X_9^{1209},$

$x_2 \cdot X_9^{1608} = X_9^{1488} + X_9^{1607},$ $\quad x_2 \cdot X_9^{1615} = X_9^{1495} + X_9^{1607},$

$x_2 \cdot X_9^{1630} = X_9^{1510} + X_9^{1627} + X_9^{1629},$ $\quad x_2 \cdot X_9^{1635} = X_9^{1515} + X_9^{1627} + X_9^{1634},$

$x_2 \cdot X_9^{1637} = X_9^{1517} + X_9^{1629} + X_9^{1634},$ $\quad x_2 \cdot X_9^{1642} = X_9^{1522} + 2X_9^{1634} + X_9^{1282},$

$x_2 \cdot X_9^{1668} = X_9^{1548} + X_9^{1660} + X_9^{1665},$ $\quad x_2 \cdot X_9^{1673} = X_9^{1553} + 2X_9^{1665} + X_9^{1329},$

$x_2 \cdot X_9^{1745} = X_9^{1742},$ $\quad x_2 \cdot X_9^{1754} = 2X_9^{1751} + X_9^{91} - X_9^{1050},$

$x_2 \cdot X_9^{1761} = X_9^{1751} + X_9^{1759},$ $\quad x_2 \cdot X_9^{1772} = X_9^{1742},$

$x_2 \cdot X_9^{1781} = X_9^{1751} + X_9^{1779},$ $\quad x_2 \cdot X_9^{1789} = X_9^{1759} + X_9^{1779},$

$x_2 \cdot X_9^{1800} = 2X_9^{1798} + X_9^{325},$ $\quad x_2 \cdot X_9^{1808} = X_9^{1798} + X_9^{1807},$

$x_2 \cdot X_9^{1834} = X_9^{1828},$ $\quad x_2 \cdot X_9^{1845} = X_9^{1839} + X_9^{1843},$

$x_2 \cdot X_9^{1849} = 2X_9^{1843} + X_9^{38} - X_9^{1073} + X_9^{1742},$

$x_2 \cdot X_9^{1862} = X_9^{1856} + 2X_9^{1860} + X_9^{559},$

$x_2 \cdot X_9^{1866} = 2X_9^{1860} + X_9^{74} - X_9^{1066} + X_9^{57} + X_9^{1751} + X_9^{1865},$

$x_2 \cdot X_9^{1871} = 3X_9^{1865} + 6X_9^{110} - 3X_9^{1059} + 3X_9^{1759} - 2X_9^{1807},$

$x_2 \cdot X_9^{1888} = X_9^{1828},$ $\quad x_2 \cdot X_9^{1899} = X_9^{1839} + X_9^{1897},$

$x_2 \cdot X_9^{1903} = X_9^{1843} + X_9^{1897},$ $\quad x_2 \cdot X_9^{1916} = X_9^{1856} + 2X_9^{1914} + X_9^{721},$

$x_2 \cdot X_9^{1920} = X_9^{1860} + X_9^{1914} + X_9^{1919},$

$x_2 \cdot X_9^{1925} = X_9^{1865} + 2X_9^{1919} + 2X_9^{218} - X_9^{1061} + X_9^{1779},$

$x_2 \cdot X_9^{1987} = 2X_9^{1984} + X_9^{1828},$

$x_2 \cdot X_9^{1998} = 2X_9^{1995} + X_9^{489} + X_9^{1839} + X_9^{1997},$

$x_2 \cdot X_9^{2000} = 3X_9^{1997} - 2X_9^{383} + 2X_9^{1128} + 3X_9^{506} + 3X_9^{1843},$

$x_2 \cdot X_9^{2047} = X_9^{1984} + X_9^{2044},$ $\quad x_2 \cdot X_9^{2058} = X_9^{1995} + X_9^{2055} + X_9^{2057},$

$x_2 \cdot X_9^{2060} = X_9^{1997} + 2X_9^{2057} + X_9^{668} + X_9^{1897},$

$x_2 \cdot X_9^{2123} = 3X_9^{2122} + 2X_9^{1488} + 3X_9^{1984},$

$x_2 \cdot X_9^{2158} = X_9^{2122} + 2X_9^{2157} + X_9^{2044}.$

Appendix F

A basis for the invariant space of degree 12 in the natural representation of sl(3)

$$\begin{aligned}
I_{12}^1 = {} & I_9^1 - X_9^{64} - X_9^{69} + X_9^{99} + X_9^{118} - X_9^{120} + X_9^{134} - X_9^{142} - X_9^{146} - X_9^{147} - X_9^{162} + X_9^{177} \\
& + X_9^{197} - X_9^{219} + X_9^{237} + X_9^{260} - X_9^{279} + X_9^{284} + X_9^{302} - X_9^{305} - X_9^{307} - X_9^{326} + X_9^{355},
\end{aligned}$$

$$\begin{aligned}
I_{12}^2 = {} & I_9^2 + X_9^{399} - 2X_9^{402} + X_9^{404} + X_9^{424} - 2X_9^{426} + X_9^{429} + X_9^{447} + X_9^{454} - 2X_9^{457} + X_9^{478} \\
& + X_9^{480} - X_9^{490} - 2X_9^{508} + X_9^{515} + X_9^{518} - X_9^{534} - 2X_9^{538} + X_9^{539} + X_9^{564} + X_9^{569} \\
& - X_9^{577} - 2X_9^{597} - X_9^{600} + X_9^{602},
\end{aligned}$$

$$\begin{aligned}
I_{12}^3 = {} & I_9^3 - X_9^{399} + 2X_9^{402} - X_9^{404} - X_9^{424} + 2X_9^{426} - X_9^{429} - X_9^{447} + 2X_9^{454} - X_9^{457} - X_9^{478} \\
& - X_9^{480} + X_9^{490} + X_9^{508} - 2X_9^{515} + X_9^{518} - X_9^{534} - 2X_9^{538} + X_9^{539} + X_9^{724} - 2X_9^{727} \\
& + X_9^{729} - X_9^{801} + 2X_9^{804} - X_9^{806} + X_9^{833} + X_9^{835} - X_9^{845} + X_9^{882} - 2X_9^{885} + X_9^{887} - X_9^{958} \\
& - X_9^{965} + 2X_9^{968} - X_9^{1048} - X_9^{1050} + X_9^{1060} + X_9^{1141} + X_9^{1143} - X_9^{1153} + X_9^{1188} + X_9^{1195} \\
& - 2X_9^{1198} + X_9^{1231} + 2X_9^{1235} - X_9^{1236} - X_9^{1287} - X_9^{1294} + 2X_9^{1297} + 2X_9^{1380} + X_9^{1383} \\
& - X_9^{1385} + X_9^{1558} - 2X_9^{1560} + X_9^{1563} - X_9^{1671} + 2X_9^{1673} - X_9^{1676} - X_9^{1723} + 2X_9^{1730} \\
& - X_9^{1733} + X_9^{1788} - 2X_9^{1790} + X_9^{1793} + X_9^{1894} + X_9^{1899} - X_9^{1907} + X_9^{2028} - 2X_9^{2035} \\
& + X_9^{2038} - X_9^{2086} - 2X_9^{2090} + X_9^{2091} - X_9^{2157} + 2X_9^{2164} - X_9^{2167} - X_9^{2214} - X_9^{2219} \\
& + X_9^{2227} - 2X_9^{2280} - X_9^{2283} + X_9^{2285} + X_9^{2349} + X_9^{2354} - X_9^{2362} + X_9^{2634} + 2X_9^{2638} \\
& - X_9^{2639} - X_9^{2781} - 2X_9^{2785} + X_9^{2786} + 2X_9^{2852} + X_9^{2855} - X_9^{2857} - 2X_9^{3005} - X_9^{3008} + X_9^{3010},
\end{aligned}$$

$$\begin{aligned}
I_{12}^4 = {} & I_9^4 - 2X_9^{527} - 2X_9^{558} - X_9^{593} + X_9^{624} + X_9^{739} - X_9^{816} - X_9^{855} + X_9^{897} - 2X_9^{977} + X_9^{979} \\
& + X_9^{1070} - X_9^{1163} + 2X_9^{1207} - X_9^{1209} + X_9^{1255} - 2X_9^{1306} + X_9^{1308} - X_9^{1407} - X_9^{1574} \\
& + X_9^{1687} + X_9^{1742} + X_9^{1744} - X_9^{1804} - X_9^{1923} - X_9^{2047} - X_9^{2049} - X_9^{2110} + X_9^{2176} + X_9^{2178} \\
& + X_9^{2243} + X_9^{2307} - X_9^{2378} + X_9^{2658} - X_9^{2805} - X_9^{2879} + X_9^{3032},
\end{aligned}$$

$$\begin{aligned}
I_{12}^5 = {} & I_9^5 + X_9^{399} + X_9^{402} - 2X_9^{404} - 2X_9^{424} + X_9^{426} + X_9^{429} + X_9^{454} - X_9^{457} - X_9^{478} + 2X_9^{480} \\
& + X_9^{490} + 2X_9^{505} - X_9^{508} - X_9^{515} - X_9^{534} - X_9^{538} + X_9^{539} + X_9^{564} - X_9^{597} - X_9^{694} \\
& - X_9^{727} + X_9^{729} - X_9^{801} + X_9^{806} + X_9^{833} - X_9^{835} - X_9^{845} - X_9^{885} + X_9^{887} + 2X_9^{921} \\
& - X_9^{923} - X_9^{926} - X_9^{955} + X_9^{968} - X_9^{1050} + X_9^{1095} + X_9^{1141} - X_9^{1143} - X_9^{1153} - 2X_9^{1185} \\
& + X_9^{1188} + X_9^{1195} + X_9^{1231} - X_9^{1236} - X_9^{1284} + X_9^{1297} - 2X_9^{1329} + X_9^{1334} - X_9^{1342} + X_9^{1380} \\
& + X_9^{1459} + X_9^{1558} - X_9^{1560} + X_9^{1617} + X_9^{1620} - 2X_9^{1622} - X_9^{1671} + X_9^{1673} - X_9^{1720} + X_9^{1730} \\
& - X_9^{1788} + X_9^{1793} - X_9^{1894} + X_9^{1899} - X_9^{1907} - X_9^{1966} + 2X_9^{1968} + X_9^{1978} + X_9^{2025} - X_9^{2035} \\
& - X_9^{2090} + 2X_9^{2154} - X_9^{2157} - X_9^{2167} + X_9^{2214} - X_9^{2219} + X_9^{2227} - X_9^{2283} + X_9^{2285} + X_9^{2349} \\
& - X_9^{2486} - X_9^{2570} + X_9^{2638} - X_9^{2781} + X_9^{2786} + X_9^{2855} - X_9^{2857} - X_9^{3005} + X_9^{3082},
\end{aligned}$$

$$\begin{aligned}
I_{12}^6 = {} & I_9^6 + X_9^{6775} - X_9^{6781} + X_9^{6786} - X_9^{6788} - X_9^{6871} + X_9^{6877} - X_9^{6882} + X_9^{6884} - X_9^{6970} \\
& + X_9^{6972} - X_9^{6974} + X_9^{7009} - X_9^{7011} + X_9^{7013} + X_9^{7207} - X_9^{7213} + X_9^{7218} - X_9^{7220} + X_9^{7258}
\end{aligned}$$

$$
\begin{aligned}
& - X_9^{7260} + X_9^{7262} - X_9^{7297} + X_9^{7299} - X_9^{7301} - X_9^{7447} + X_9^{7453} - X_9^{7458} + X_9^{7460} - X_9^{7498} \\
& + X_9^{7500} - X_9^{7502} + X_9^{7537} - X_9^{7539} + X_9^{7541},
\end{aligned}
$$

$$
\begin{aligned}
I_{12}^{7} = \; & I_9^{7} + X_9^{6736} - X_9^{6741} - X_9^{6781} + X_9^{6786} - X_9^{6871} + X_9^{6884} + X_9^{6916} - X_9^{6929} + X_9^{6966} \\
& - X_9^{6974} - X_9^{7011} + X_9^{7019} - X_9^{7072} + X_9^{7077} + 2X_9^{7158} + X_9^{7162} - X_9^{7164} - X_9^{7166} \\
& + X_9^{7207} - X_9^{7220} - X_9^{7254} + X_9^{7262} - X_9^{7297} - X_9^{7301} + X_9^{7307} + X_9^{7392} - X_9^{7400} + X_9^{7453} \\
& - X_9^{7458} - X_9^{7494} - X_9^{7498} + X_9^{7500} - X_9^{7539} + X_9^{7547} - X_9^{7588} + X_9^{7601} + X_9^{7633} + X_9^{7635} \\
& + X_9^{7637} - 2X_9^{7643} - X_9^{7728} + X_9^{7736},
\end{aligned}
$$

$$
\begin{aligned}
I_{12}^{8} = \; & I_9^{8} + X_9^{6694} - X_9^{6741} - X_9^{6786} + 2X_9^{6788} - X_9^{6835} + 2X_9^{6882} - X_9^{6884} - X_9^{6929} + X_9^{6966} \\
& + X_9^{6970} - 2X_9^{6972} + X_9^{6974} - X_9^{7009} + X_9^{7011} - 2X_9^{7013} + X_9^{7019} - X_9^{7072} + X_9^{7119} \\
& + X_9^{7162} - X_9^{7166} - X_9^{7207} + 2X_9^{7213} - X_9^{7218} + X_9^{7220} - 2X_9^{7258} - X_9^{7262} + X_9^{7297} - X_9^{7299} \\
& + X_9^{7301} + X_9^{7307} + X_9^{7353} - X_9^{7400} + 2X_9^{7447} - X_9^{7453} + X_9^{7458} - X_9^{7460} - X_9^{7494} + X_9^{7498} \\
& - X_9^{7500} + X_9^{7502} - 2X_9^{7537} + X_9^{7539} - X_9^{7588} + X_9^{7633} + X_9^{7635} - X_9^{7682} - X_9^{7728} + X_9^{7775},
\end{aligned}
$$

$$
\begin{aligned}
I_{12}^{9} = \; & I_9^{9} + 3X_9^{6775} - 3X_9^{6781} + 3X_9^{6786} - 3X_9^{6788} - 3X_9^{6871} + 3X_9^{6877} - 3X_9^{6882} + 3X_9^{6884} \\
& + 2X_9^{7162} - 2X_9^{7164} + 2X_9^{7166} + 2X_9^{7207} - 2X_9^{7213} + 2X_9^{7218} - 2X_9^{7220} - X_9^{7258} + X_9^{7260} \\
& - X_9^{7262} - X_9^{7447} + X_9^{7453} - X_9^{7458} + X_9^{7460} - X_9^{7498} + X_9^{7500} - X_9^{7502} - X_9^{8167} + X_9^{8173} \\
& - X_9^{8178} + X_9^{8180} + X_9^{8455} - X_9^{8461} + X_9^{8466} - X_9^{8468} - X_9^{8554} + X_9^{8556} - X_9^{8558} \\
& - 2X_9^{8695} + 2X_9^{8701} - 2X_9^{8706} + 2X_9^{8708} + X_9^{8791} - X_9^{8797} + X_9^{8802} - X_9^{8804} + X_9^{8929} \\
& - X_9^{8931} + X_9^{8933} + X_9^{9226} - X_9^{9228} + X_9^{9230} - 2X_9^{9466} + 2X_9^{9468} - 2X_9^{9470} - X_9^{9511} \\
& + X_9^{9517} - X_9^{9522} + X_9^{9524} + X_9^{9562} - X_9^{9564} + X_9^{9566} - X_9^{9601} + X_9^{9603} - X_9^{9605} + X_9^{9751} \\
& - X_9^{9757} + X_9^{9762} - X_9^{9764} + 2X_9^{9841} - 2X_9^{9843} + 2X_9^{9845} - X_9^{9937} + X_9^{9939} - X_9^{9941} \\
& + X_9^{10423} - X_9^{10429} + X_9^{10434} - X_9^{10436} + X_9^{10615} - X_9^{10621} + X_9^{10626} - X_9^{10628} - 2X_9^{10711} \\
& + 2X_9^{10717} - 2X_9^{10722} + 2X_9^{10724} + X_9^{10810} - X_9^{10812} + X_9^{10814} + X_9^{10951} - X_9^{10957} \\
& + X_9^{10962} - X_9^{10964} - X_9^{11185} + X_9^{11187} - X_9^{11189} + X_9^{11386} - X_9^{11388} + X_9^{11390} + X_9^{11431} \\
& - X_9^{11437} + X_9^{11442} - X_9^{11444} - 2X_9^{11482} + 2X_9^{11484} - 2X_9^{11486} - X_9^{11671} + X_9^{11677} \\
& - X_9^{11682} + X_9^{11684} + X_9^{11722} - X_9^{11724} + X_9^{11726} - X_9^{11761} + X_9^{11763} - X_9^{11765} + 2X_9^{11857} \\
& - 2X_9^{11859} + 2X_9^{11861} - X_9^{12097} + X_9^{12099} - X_9^{12101} + X_9^{12586} - X_9^{12588} + X_9^{12590} - X_9^{12826} \\
& + X_9^{12828} - X_9^{12830} - X_9^{12961} + X_9^{12963} - X_9^{12965} + X_9^{13201} - X_9^{13203} + X_9^{13205},
\end{aligned}
$$

$$
\begin{aligned}
I_{12}^{10} = \; & I_9^{10} - X_9^{6736} + X_9^{6741} - X_9^{6775} + 2X_9^{6781} - 2X_9^{6786} + X_9^{6788} + 2X_9^{6871} - X_9^{6877} + X_9^{6882} \\
& - 2X_9^{6884} - X_9^{6916} + X_9^{6929} + 2X_9^{6966} + X_9^{6970} - X_9^{6972} - X_9^{6974} - X_9^{7009} - X_9^{7011} \\
& - X_9^{7013} + 2X_9^{7019} + X_9^{7072} - X_9^{7077} - X_9^{7162} + X_9^{7164} - X_9^{7166} - X_9^{7207} + X_9^{7213} - X_9^{7218} \\
& + X_9^{7220} - 3X_9^{7254} - X_9^{7258} + X_9^{7260} + 2X_9^{7262} + X_9^{7297} + X_9^{7299} + X_9^{7301} - 2X_9^{7307} \\
& + X_9^{7392} - X_9^{7400} - X_9^{7453} + X_9^{7458} + X_9^{7494} + X_9^{7498} - X_9^{7500} + X_9^{8128} - X_9^{8133} - X_9^{8173} \\
& + X_9^{8178} + X_9^{8320} - X_9^{8325} - 2X_9^{8416} + 2X_9^{8421} + X_9^{8461} - X_9^{8466} + X_9^{8550} + X_9^{8554} - X_9^{8556} \\
& + X_9^{8656} - X_9^{8661} - 2X_9^{8701} + 2X_9^{8706} + X_9^{8797} - X_9^{8802} - 2X_9^{8886} - X_9^{8890} + X_9^{8892} \\
& + X_9^{8894} + X_9^{8931} - X_9^{8939} + X_9^{9136} - X_9^{9141} - X_9^{9222} - X_9^{9226} + X_9^{9228} - X_9^{9376} + X_9^{9381} \\
& + X_9^{9466} - X_9^{9468} + X_9^{9470} - X_9^{9517} + X_9^{9522} + 3X_9^{9558} + X_9^{9562} - X_9^{9564} - 2X_9^{9566} \\
& - X_9^{9603} + X_9^{9611} - X_9^{9696} + X_9^{9704} + X_9^{9757} - X_9^{9762} - 2X_9^{9798} - X_9^{9802} + X_9^{9804} + X_9^{9806} \\
& + 2X_9^{9843} - 2X_9^{9851} - X_9^{9939} + X_9^{9947} + X_9^{10032} - X_9^{10040} - X_9^{10423} + X_9^{10436} + X_9^{10468} \\
& - X_9^{10481} - X_9^{10615} + X_9^{10628} + 2X_9^{10711} - 2X_9^{10724} - X_9^{10756} + X_9^{10769} + X_9^{10806} - X_9^{10814} \\
& - X_9^{10849} - X_9^{10851} - X_9^{10853} + 2X_9^{10859} - X_9^{10951} + X_9^{10964} + 2X_9^{10996} - 2X_9^{11009} \\
& - X_9^{11092} + X_9^{11105} + X_9^{11185} + X_9^{11189} - X_9^{11195} + X_9^{11382} - X_9^{11390} - X_9^{11431} + X_9^{11444} \\
& - 2X_9^{11478} + 2X_9^{11486} + X_9^{11521} + X_9^{11523} + X_9^{11525} - 2X_9^{11531} + X_9^{11616} - X_9^{11624}
\end{aligned}
$$

$$
\begin{aligned}
&+X_9^{11671}-X_9^{11684}+X_9^{11718}-X_9^{11726}-X_9^{11761}-2X_9^{11763}-X_9^{11765}+3X_9^{11771}+X_9^{11812}\\
&-X_9^{11825}-X_9^{11857}+X_9^{11859}-X_9^{11861}-X_9^{11952}+X_9^{11960}-X_9^{12052}+X_9^{12065}+X_9^{12097}\\
&+X_9^{12101}-X_9^{12107}+X_9^{12582}-X_9^{12590}-X_9^{12672}+X_9^{12680}-X_9^{12822}+X_9^{12830}+X_9^{12912}\\
&-X_9^{12920}-X_9^{12963}+X_9^{12971}+X_9^{13008}-X_9^{13016}+X_9^{13203}-X_9^{13211}-X_9^{13248}+X_9^{13256},
\end{aligned}
$$

$$
\begin{aligned}
I_{12}^{11} = {} & I_9^{11}-X_9^{6694}+X_9^{6741}+3X_9^{6775}-3X_9^{6781}+X_9^{6786}-2X_9^{6788}+X_9^{6835}-3X_9^{6871}\\
&+3X_9^{6877}-2X_9^{6882}+X_9^{6884}+X_9^{6929}+2X_9^{6966}-X_9^{6970}-X_9^{6972}-X_9^{6974}+X_9^{7009}\\
&-X_9^{7011}-X_9^{7013}+2X_9^{7019}+X_9^{7072}-X_9^{7119}+X_9^{7162}+X_9^{7166}+3X_9^{7207}-3X_9^{7213}\\
&+X_9^{7218}-X_9^{7220}-2X_9^{7254}+X_9^{7260}+X_9^{7262}-X_9^{7297}+X_9^{7299}+X_9^{7301}-2X_9^{7307}\\
&+X_9^{7353}-X_9^{7400}-2X_9^{7447}+X_9^{7453}-X_9^{7458}+X_9^{7460}+X_9^{7494}-X_9^{7498}+X_9^{7500}-X_9^{7502}\\
&+X_9^{7537}-X_9^{7539}-X_9^{7942}+X_9^{8038}-X_9^{8133}+X_9^{8180}+X_9^{8320}-X_9^{8374}-X_9^{8416}+X_9^{8421}\\
&-X_9^{8468}+X_9^{8511}-X_9^{8558}+X_9^{8614}-X_9^{8661}-2X_9^{8695}+X_9^{8701}+X_9^{8708}+2X_9^{8791}-X_9^{8797}\\
&-2X_9^{8886}+X_9^{8892}+2X_9^{8933}-X_9^{8939}+X_9^{9136}-X_9^{9183}+X_9^{9230}-X_9^{9376}+X_9^{9423}-X_9^{9466}\\
&-X_9^{9470}-2X_9^{9511}+X_9^{9517}+2X_9^{9558}+X_9^{9562}-X_9^{9564}-2X_9^{9605}+X_9^{9611}-X_9^{9657}\\
&+X_9^{9704}+2X_9^{9751}-X_9^{9757}-2X_9^{9798}+X_9^{9804}+X_9^{9841}+2X_9^{9845}-X_9^{9851}-X_9^{9937}\\
&+X_9^{10032}-X_9^{10079}+X_9^{10243}-X_9^{10339}+X_9^{10434}-X_9^{10481}+X_9^{10615}-2X_9^{10621}+X_9^{10675}\\
&-X_9^{10711}+2X_9^{10717}-X_9^{10722}+X_9^{10769}+X_9^{10806}-2X_9^{10812}-X_9^{10853}+2X_9^{10859}\\
&-X_9^{10915}+X_9^{10962}+X_9^{10996}-X_9^{11009}-X_9^{11092}+X_9^{11187}-X_9^{11234}+X_9^{11386}+X_9^{11431}\\
&-2X_9^{11437}-X_9^{11478}-X_9^{11482}+2X_9^{11484}+X_9^{11525}-2X_9^{11531}+X_9^{11577}-X_9^{11624}-X_9^{11671}\\
&+2X_9^{11677}+X_9^{11718}-2X_9^{11724}-X_9^{11761}-X_9^{11765}+2X_9^{11771}+X_9^{11812}+X_9^{11857}\\
&-X_9^{11859}+X_9^{11906}-X_9^{11952}+X_9^{11999}-X_9^{12052}+X_9^{12099}-X_9^{12146}+X_9^{12586}-X_9^{12633}\\
&+X_9^{12680}-X_9^{12826}+X_9^{12873}-X_9^{12920}-X_9^{12961}+X_9^{13008}-X_9^{13055}+X_9^{13201}-X_9^{13248}\\
&+X_9^{13295},
\end{aligned}
$$

$$
\begin{aligned}
I_{12}^{12} = {} & I_9^{12}+X_9^{6694}+X_9^{6736}+X_9^{6741}+X_9^{6775}-2X_9^{6781}-2X_9^{6786}+X_9^{6788}-X_9^{6835}-2X_9^{6871}\\
&+X_9^{6877}+X_9^{6882}-2X_9^{6884}+X_9^{6916}+X_9^{6929}+2X_9^{6966}-X_9^{6972}-X_9^{6974}-X_9^{7011}\\
&-X_9^{7013}+2X_9^{7019}-X_9^{7072}-X_9^{7077}-X_9^{7119}+X_9^{7162}+3X_9^{7164}-X_9^{7166}+X_9^{7207}\\
&+X_9^{7213}+X_9^{7220}-X_9^{7254}-X_9^{7258}-X_9^{7260}+2X_9^{7262}-X_9^{7297}+X_9^{7301}-2X_9^{7307}\\
&+X_9^{7353}+X_9^{7392}-X_9^{7400}-X_9^{7539}-X_9^{7588}+X_9^{7633}+X_9^{7635}-X_9^{7728}+X_9^{8038}-X_9^{8083}\\
&-X_9^{8133}+X_9^{8178}-X_9^{8325}-X_9^{8374}-X_9^{8416}+X_9^{8421}-X_9^{8455}+2X_9^{8461}+X_9^{8511}+X_9^{8550}\\
&-2X_9^{8556}-X_9^{8661}+2X_9^{8706}+X_9^{8755}+2X_9^{8791}-X_9^{8797}-X_9^{8802}+X_9^{8804}-X_9^{8836}\\
&-X_9^{8849}-2X_9^{8886}+X_9^{8892}+X_9^{8931}+X_9^{9087}+X_9^{9136}-X_9^{9183}-X_9^{9226}+X_9^{9321}+X_9^{9381}\\
&+X_9^{9423}-3X_9^{9468}-X_9^{9511}-X_9^{9517}-X_9^{9524}+X_9^{9558}+X_9^{9562}+X_9^{9564}-X_9^{9566}+X_9^{9601}\\
&-X_9^{9605}+2X_9^{9611}-X_9^{9657}-X_9^{9696}-X_9^{9762}-2X_9^{9798}+X_9^{9804}+2X_9^{9843}+X_9^{9892}\\
&+X_9^{9905}-X_9^{9937}-X_9^{9939}+2X_9^{9941}-X_9^{9947}-X_9^{9986}+X_9^{10032}-X_9^{10198}+X_9^{10243}\\
&+X_9^{10436}-X_9^{10481}+X_9^{10534}+X_9^{10576}+X_9^{10581}+X_9^{10615}-2X_9^{10621}-X_9^{10626}+X_9^{10628}\\
&-2X_9^{10724}+X_9^{10769}-X_9^{10814}-X_9^{10853}+2X_9^{10859}-X_9^{10915}-2X_9^{10951}+X_9^{10957}\\
&+X_9^{10996}-X_9^{11009}+X_9^{11105}+2X_9^{11189}-X_9^{11195}-X_9^{11234}-X_9^{11296}-X_9^{11301}-X_9^{11343}\\
&-X_9^{11382}+X_9^{11386}+2X_9^{11388}-X_9^{11390}+X_9^{11444}+2X_9^{11486}+X_9^{11525}-2X_9^{11531}-X_9^{11624}\\
&+X_9^{11671}+X_9^{11677}+X_9^{11682}+2X_9^{11718}-X_9^{11722}-X_9^{11724}-X_9^{11761}-X_9^{11763}+X_9^{11765}\\
&+X_9^{11771}-X_9^{11825}-3X_9^{11861}+X_9^{11906}+X_9^{11960}+X_9^{11999}-X_9^{12052}+X_9^{12097}-X_9^{12146}\\
&+X_9^{12242}-X_9^{12335}+X_9^{12447}-X_9^{12537}-X_9^{12590}+X_9^{12680}-X_9^{12822}-X_9^{12828}+X_9^{12873}\\
&+X_9^{12912}+X_9^{12965}+X_9^{12971}-X_9^{13016}-X_9^{13055}+X_9^{13203}-X_9^{13248}-X_9^{13346}+X_9^{13391},
\end{aligned}
$$

$$
\begin{aligned}
I_{12}^{13} = {} & I_9^{13}+2X_9^{6694}+X_9^{6736}+3X_9^{6752}-3X_9^{6761}-5X_9^{6775}+X_9^{6781}-3X_9^{6786}+6X_9^{6788}\\
&-3X_9^{6799}+3X_9^{6808}-2X_9^{6835}-5X_9^{6871}+X_9^{6877}+3X_9^{6882}-3X_9^{6895}+3X_9^{6904}+7X_9^{6916}
\end{aligned}
$$

$$-3X_9^{6929}+3X_9^{6942}-3X_9^{6951}+9X_9^{6966}+X_9^{6970}-3X_9^{6972}-6X_9^{6974}+3X_9^{6988}-3X_9^{6997}$$
$$-4X_9^{7009}-6X_9^{7011}-3X_9^{7013}+9X_9^{7019}-3X_9^{7035}+3X_9^{7044}-2X_9^{7072}-X_9^{7077}-2X_9^{7088}$$
$$+X_9^{7097}+2X_9^{7142}+8X_9^{7158}+2X_9^{7164}-8X_9^{7166}+4X_9^{7180}-4X_9^{7189}+2X_9^{7207}+3X_9^{7213}$$
$$-X_9^{7218}+2X_9^{7231}-X_9^{7240}-4X_9^{7254}-3X_9^{7258}-X_9^{7260}+4X_9^{7262}-2X_9^{7276}-X_9^{7285}$$
$$-2X_9^{7297}-4X_9^{7299}+3X_9^{7307}-2X_9^{7323}+3X_9^{7332}+2X_9^{7353}+2X_9^{7379}+6X_9^{7392}-6X_9^{7400}$$
$$+2X_9^{7417}-3X_9^{7426}+2X_9^{7447}-X_9^{7453}+X_9^{7458}-X_9^{7460}+X_9^{7471}-X_9^{7494}+X_9^{7498}-X_9^{7500}$$
$$+X_9^{7502}-X_9^{7516}-X_9^{7537}-X_9^{7539}-X_9^{7563}-2X_9^{7588}-X_9^{7614}+2X_9^{7633}+2X_9^{7635}+2X_9^{7659}$$
$$-2X_9^{7728}-X_9^{7753}-X_9^{7918}+X_9^{7942}+X_9^{7965}+X_9^{8038}+X_9^{8061}-2X_9^{8083}-X_9^{8108}+X_9^{8128}$$
$$-2X_9^{8133}+X_9^{8144}-X_9^{8153}-2X_9^{8167}+2X_9^{8178}+X_9^{8180}-X_9^{8191}+X_9^{8200}+X_9^{8254}+X_9^{8298}$$
$$-3X_9^{8325}+2X_9^{8336}-3X_9^{8345}-X_9^{8374}-X_9^{8397}-3X_9^{8416}+3X_9^{8421}-X_9^{8432}-3X_9^{8455}$$
$$+4X_9^{8461}+2X_9^{8468}-X_9^{8479}+2X_9^{8488}+X_9^{8511}+X_9^{8534}+5X_9^{8550}-4X_9^{8556}-2X_9^{8558}$$
$$+X_9^{8572}-2X_9^{8581}-X_9^{8614}-X_9^{8637}-X_9^{8656}+X_9^{8661}-3X_9^{8672}+2X_9^{8681}+2X_9^{8695}-X_9^{8701}$$
$$+3X_9^{8706}-X_9^{8708}+X_9^{8719}+2X_9^{8755}+X_9^{8780}+8X_9^{8791}-X_9^{8797}-3X_9^{8802}-4X_9^{8804}$$
$$+4X_9^{8815}-3X_9^{8824}-5X_9^{8836}+X_9^{8849}-2X_9^{8862}+X_9^{8871}-9X_9^{8886}-X_9^{8890}+3X_9^{8892}$$
$$+5X_9^{8894}-3X_9^{8908}+2X_9^{8917}+2X_9^{8929}+4X_9^{8931}-3X_9^{8939}+2X_9^{8955}-X_9^{8964}-X_9^{9018}$$
$$+2X_9^{9087}+2X_9^{9110}+2X_9^{9136}-X_9^{9141}+X_9^{9161}-X_9^{9183}-X_9^{9206}-2X_9^{9226}-X_9^{9230}-X_9^{9253}$$
$$+2X_9^{9321}+X_9^{9347}+2X_9^{9376}+2X_9^{9392}-2X_9^{9446}-8X_9^{9462}-2X_9^{9468}+6X_9^{9470}-4X_9^{9484}$$
$$+2X_9^{9493}-3X_9^{9511}-3X_9^{9517}+X_9^{9522}+X_9^{9524}-2X_9^{9535}+4X_9^{9558}+3X_9^{9562}+X_9^{9564}$$
$$-3X_9^{9566}+2X_9^{9580}+2X_9^{9589}+3X_9^{9601}+3X_9^{9603}+X_9^{9605}-2X_9^{9611}+2X_9^{9627}-2X_9^{9636}$$
$$-2X_9^{9657}-2X_9^{9683}-6X_9^{9696}+4X_9^{9704}-2X_9^{9721}+2X_9^{9730}-4X_9^{9751}+X_9^{9757}-X_9^{9762}$$
$$+2X_9^{9764}-2X_9^{9775}+X_9^{9784}+3X_9^{9798}-3X_9^{9806}+3X_9^{9820}-X_9^{9829}+X_9^{9841}+5X_9^{9843}$$
$$-3X_9^{9851}+X_9^{9867}-X_9^{9876}+5X_9^{9892}-X_9^{9905}+2X_9^{9918}-X_9^{9927}-5X_9^{9937}-7X_9^{9939}$$
$$+6X_9^{9947}-5X_9^{9963}+2X_9^{9972}-X_9^{9986}+X_9^{10010}+6X_9^{10032}-4X_9^{10040}+3X_9^{10057}-X_9^{10066}$$
$$-X_9^{10104}-X_9^{10198}+X_9^{10221}-X_9^{10268}-X_9^{10364}+X_9^{10411}-2X_9^{10423}+3X_9^{10436}-X_9^{10447}$$
$$+X_9^{10456}+3X_9^{10468}-3X_9^{10481}+X_9^{10494}-X_9^{10503}+X_9^{10534}-X_9^{10557}+X_9^{10576}+X_9^{10581}$$
$$+X_9^{10592}-2X_9^{10601}-5X_9^{10615}+6X_9^{10628}-3X_9^{10639}+4X_9^{10648}+X_9^{10700}+X_9^{10711}$$
$$-3X_9^{10724}+X_9^{10744}+5X_9^{10756}-3X_9^{10769}+2X_9^{10782}-3X_9^{10791}+3X_9^{10806}-4X_9^{10814}$$
$$+X_9^{10828}-2X_9^{10837}-3X_9^{10849}-5X_9^{10851}-3X_9^{10853}+9X_9^{10859}-2X_9^{10875}+3X_9^{10884}$$
$$+X_9^{10940}+X_9^{10957}-X_9^{10962}-2X_9^{10964}+2X_9^{10975}-X_9^{10984}+3X_9^{10996}-3X_9^{11009}-X_9^{11031}$$
$$-X_9^{11083}-6X_9^{11092}+6X_9^{11105}-3X_9^{11118}+2X_9^{11127}+X_9^{11165}+2X_9^{11185}+2X_9^{11187}$$
$$+4X_9^{11189}-5X_9^{11195}+2X_9^{11211}-X_9^{11220}-X_9^{11234}-X_9^{11258}-X_9^{11296}-X_9^{11301}-X_9^{11312}$$
$$+2X_9^{11321}-X_9^{11343}+X_9^{11366}+6X_9^{11382}-7X_9^{11390}+2X_9^{11404}-5X_9^{11413}+X_9^{11431}$$
$$+X_9^{11455}-2X_9^{11464}-3X_9^{11478}+5X_9^{11486}-X_9^{11500}+X_9^{11509}-X_9^{11521}-3X_9^{11523}$$
$$+3X_9^{11531}-X_9^{11547}+3X_9^{11556}+X_9^{11603}+4X_9^{11616}-6X_9^{11624}+X_9^{11641}-3X_9^{11650}$$
$$+3X_9^{11671}-X_9^{11677}+X_9^{11682}-X_9^{11684}-2X_9^{11704}-2X_9^{11718}-X_9^{11722}+X_9^{11724}$$
$$+3X_9^{11726}-2X_9^{11740}+2X_9^{11749}-X_9^{11761}-3X_9^{11763}+X_9^{11765}+4X_9^{11771}+2X_9^{11787}$$
$$+2X_9^{11796}-2X_9^{11812}+2X_9^{11847}+2X_9^{11857}+6X_9^{11859}-2X_9^{11861}-8X_9^{11867}+2X_9^{11883}$$
$$-4X_9^{11892}-2X_9^{11930}-4X_9^{11952}+6X_9^{11960}-2X_9^{11977}+2X_9^{11986}+2X_9^{11999}+2X_9^{12024}$$
$$+X_9^{12078}+X_9^{12097}-X_9^{12099}-X_9^{12123}-X_9^{12146}-X_9^{12170}-X_9^{12221}+2X_9^{12242}+2X_9^{12266}$$
$$-2X_9^{12335}-X_9^{12360}+X_9^{12447}-X_9^{12470}+2X_9^{12563}-X_9^{12590}+X_9^{12613}-X_9^{12659}-X_9^{12706}$$
$$+X_9^{12801}-X_9^{12822}+X_9^{12826}-X_9^{12828}+X_9^{12844}+X_9^{12853}-X_9^{12899}-2X_9^{12937}-X_9^{12946}$$
$$+X_9^{12965}+X_9^{12971}-X_9^{12987}-X_9^{12996}+X_9^{13033}+2X_9^{13042}+X_9^{13080}-X_9^{13137}-X_9^{13175}$$
$$-X_9^{13201}+X_9^{13203}-X_9^{13227}+X_9^{13273}+X_9^{13320}-X_9^{13346}+X_9^{13370}-2X_9^{13416}+X_9^{13511},$$

$$I_{12}^{14}= I_9^{14}-X_9^{6775}+X_9^{6781}-X_9^{6786}+X_9^{6788}-X_9^{7009}+X_9^{7011}-X_9^{7013}-X_9^{7162}+X_9^{7164}$$

$$
\begin{aligned}
&- X_9^{7166} + X_9^{7297} - X_9^{7299} + X_9^{7301} + X_9^{7498} - X_9^{7500} + X_9^{7502} - X_9^{7633} + X_9^{7635} - X_9^{7637} \\
&+ X_9^{13783} - X_9^{13789} + X_9^{13794} - X_9^{13796} + X_9^{13834} - X_9^{13836} + X_9^{13838} - X_9^{14023} + X_9^{14029} \\
&- X_9^{14034} + X_9^{14036} - X_9^{14074} + X_9^{14076} - X_9^{14078} + X_9^{14215} - X_9^{14221} + X_9^{14226} - X_9^{14228} \\
&+ 2X_9^{14266} - 2X_9^{14268} + 2X_9^{14270} - X_9^{14353} + X_9^{14355} - X_9^{14357} - X_9^{14551} + X_9^{14557} \\
&- X_9^{14562} + X_9^{14564} - X_9^{14689} + X_9^{14691} - X_9^{14693} + X_9^{14791} - X_9^{14797} + X_9^{14802} - X_9^{14804} \\
&- X_9^{14842} + X_9^{14844} - X_9^{14846} + 2X_9^{14929} - 2X_9^{14931} + 2X_9^{14933} - X_9^{15031} + X_9^{15037} \\
&- X_9^{15042} + X_9^{15044} - X_9^{15121} + X_9^{15123} - X_9^{15125} + X_9^{15370} - X_9^{15372} + X_9^{15374} - X_9^{15457} \\
&+ X_9^{15459} - X_9^{15461} - X_9^{15610} + X_9^{15612} - X_9^{15614} + X_9^{15649} - X_9^{15651} + X_9^{15653},
\end{aligned}
$$

$$
\begin{aligned}
I_{12}^{15} = \;& I_9^{15} - X_9^{6736} + X_9^{6741} + X_9^{6871} - X_9^{6884} - X_9^{6966} + X_9^{6974} + X_9^{7072} - X_9^{7077} - X_9^{7158} \\
&+ X_9^{7166} - X_9^{7207} + X_9^{7220} + X_9^{7254} - X_9^{7262} - X_9^{7453} + X_9^{7458} + X_9^{7539} - X_9^{7547} + X_9^{7588} \\
&- X_9^{7601} - X_9^{7635} + X_9^{7643} - X_9^{13696} + X_9^{13701} + X_9^{13789} - X_9^{13794} - X_9^{13830} - X_9^{13834} \\
&+ X_9^{13836} + X_9^{13936} - X_9^{13941} - X_9^{14029} + X_9^{14034} + X_9^{14070} + X_9^{14074} - X_9^{14076} - X_9^{14176} \\
&+ X_9^{14181} + X_9^{14221} - X_9^{14226} - X_9^{14266} + X_9^{14268} - X_9^{14270} - X_9^{14355} + X_9^{14363} + X_9^{14400} \\
&- X_9^{14408} + X_9^{14551} - X_9^{14564} - X_9^{14644} + X_9^{14657} + X_9^{14689} + X_9^{14693} - X_9^{14699} - X_9^{14791} \\
&+ X_9^{14804} - X_9^{14838} + X_9^{14846} + X_9^{14884} - X_9^{14897} - X_9^{14929} + X_9^{14931} - X_9^{14933} - X_9^{14976} \\
&+ X_9^{14984} + X_9^{15031} - X_9^{15044} - X_9^{15076} + X_9^{15089} + X_9^{15121} + X_9^{15125} - X_9^{15131} + X_9^{15366} \\
&- X_9^{15374} - X_9^{15459} + X_9^{15467} + X_9^{15504} - X_9^{15512} - X_9^{15606} + X_9^{15614} + X_9^{15651} - X_9^{15659} \\
&- X_9^{15696} + X_9^{15704},
\end{aligned}
$$

$$
\begin{aligned}
I_{12}^{16} = \;& I_9^{16} + X_9^{6694} - X_9^{6736} + 2X_9^{6775} - X_9^{6781} + X_9^{6786} - X_9^{6884} + X_9^{6974} + X_9^{7009} - X_9^{7011} \\
&- X_9^{7077} + X_9^{7119} - X_9^{7158} + X_9^{7162} - X_9^{7164} + X_9^{7166} + X_9^{7220} - X_9^{7262} - X_9^{7297} + X_9^{7299} \\
&+ X_9^{7458} - X_9^{7498} - X_9^{7502} + X_9^{7539} - X_9^{7601} + X_9^{7633} - X_9^{7635} + X_9^{7637} + X_9^{7643} - X_9^{7682} \\
&- X_9^{13654} + X_9^{13701} + X_9^{13747} - 2X_9^{13794} + X_9^{13796} - X_9^{13830} - X_9^{13834} + 2X_9^{13836} \\
&- X_9^{13838} + X_9^{13936} - X_9^{13983} + X_9^{14023} - 2X_9^{14029} + X_9^{14034} - X_9^{14036} + 2X_9^{14074} \\
&+ X_9^{14078} - X_9^{14121} + X_9^{14181} - 2X_9^{14215} + X_9^{14221} - 2X_9^{14226} - 2X_9^{14266} + 3X_9^{14268} \\
&- 2X_9^{14270} + X_9^{14308} + X_9^{14321} - 2X_9^{14355} + 2X_9^{14357} - X_9^{14363} + X_9^{14400} + X_9^{14470} \\
&- X_9^{14515} + X_9^{14562} - 2X_9^{14564} + X_9^{14657} + X_9^{14689} - X_9^{14691} + 2X_9^{14693} - X_9^{14699} - X_9^{14752} \\
&- X_9^{14757} - X_9^{14791} + 2X_9^{14797} + 2X_9^{14804} + X_9^{14838} - 2X_9^{14844} + 2X_9^{14846} - X_9^{14897} \\
&- 2X_9^{14929} + 2X_9^{14931} - 3X_9^{14933} + X_9^{14984} + 2X_9^{15031} - X_9^{15037} + X_9^{15042} - X_9^{15044} \\
&- X_9^{15076} + 2X_9^{15121} - X_9^{15123} + X_9^{15218} - X_9^{15263} + X_9^{15327} - X_9^{15370} - 2X_9^{15374} + X_9^{15417} \\
&+ X_9^{15457} - X_9^{15459} + X_9^{15461} + X_9^{15467} - X_9^{15512} - X_9^{15606} + X_9^{15610} - X_9^{15612} + X_9^{15614} \\
&- X_9^{15649} + 2X_9^{15651} - X_9^{15696} - X_9^{15746} + X_9^{15791},
\end{aligned}
$$

$$
\begin{aligned}
I_{12}^{17} = \;& I_9^{17} + X_9^{6694} + X_9^{6717} - 2X_9^{6736} + X_9^{6741} - X_9^{6761} + 4X_9^{6775} - X_9^{6781} + X_9^{6786} - 2X_9^{6788} \\
&+ 2X_9^{6799} - X_9^{6808} - X_9^{6860} - X_9^{6871} - X_9^{6895} + 2X_9^{6904} - X_9^{6942} + X_9^{6974} - X_9^{6997} \\
&+ X_9^{7009} - X_9^{7011} + X_9^{7035} + X_9^{7072} - 2X_9^{7077} + X_9^{7088} - X_9^{7097} + X_9^{7119} + X_9^{7142} \\
&- 4X_9^{7158} + X_9^{7162} - X_9^{7164} + 4X_9^{7166} - 2X_9^{7180} + X_9^{7189} - 2X_9^{7207} + 3X_9^{7220} - X_9^{7231} \\
&+ X_9^{7240} + 3X_9^{7254} - 4X_9^{7262} + X_9^{7276} - 2X_9^{7285} - X_9^{7297} + X_9^{7299} + X_9^{7323} + X_9^{7379} \\
&- X_9^{7417} - 2X_9^{7447} + 2X_9^{7458} + X_9^{7460} - X_9^{7471} + X_9^{7480} - 2X_9^{7498} - X_9^{7502} - X_9^{7525} \\
&+ 2X_9^{7537} + 4X_9^{7539} - 3X_9^{7547} + 2X_9^{7563} - X_9^{7572} + 3X_9^{7588} - 3X_9^{7601} + X_9^{7614} - X_9^{7623} \\
&- 4X_9^{7635} + X_9^{7637} + 4X_9^{7643} - X_9^{7659} + 2X_9^{7668} - X_9^{7682} - X_9^{7706} - X_9^{7762} + X_9^{7800} \\
&+ X_9^{13630} - 2X_9^{13654} - 2X_9^{13677} - 2X_9^{13696} + 4X_9^{13701} - 2X_9^{13712} + 2X_9^{13721} + 2X_9^{13747} \\
&+ X_9^{13772} + 4X_9^{13783} - 4X_9^{13794} - 2X_9^{13796} + 2X_9^{13807} - 2X_9^{13816} - 5X_9^{13830} - 2X_9^{13834} \\
&+ 4X_9^{13836} + X_9^{13838} - X_9^{13852} + X_9^{13861} - X_9^{13914} + 4X_9^{13936} - 2X_9^{13941} + 2X_9^{13961} \\
&- 2X_9^{13983} - 2X_9^{14006} + X_9^{14023} - 4X_9^{14029} + 2X_9^{14034} - X_9^{14036} - X_9^{14056} + 4X_9^{14074}
\end{aligned}
$$

$$
\begin{aligned}
&+ 2X_9^{14078} + 2X_9^{14101} - 2X_9^{14121} - X_9^{14147} + X_9^{14181} + 2X_9^{14192} - X_9^{14201} - 8X_9^{14215} \\
&+ 2X_9^{14221} - 2X_9^{14226} + 4X_9^{14228} - 4X_9^{14239} + 2X_9^{14248} + 8X_9^{14262} - 2X_9^{14266} + 2X_9^{14268} \\
&- 8X_9^{14270} + 4X_9^{14284} - 2X_9^{14293} + 5X_9^{14308} - X_9^{14321} + 2X_9^{14334} - X_9^{14343} - 4X_9^{14353} \\
&- 8X_9^{14355} + 6X_9^{14363} - 4X_9^{14379} + 2X_9^{14388} + 6X_9^{14400} - 4X_9^{14408} + 2X_9^{14425} - X_9^{14434} \\
&+ X_9^{14470} - X_9^{14493} + 2X_9^{14540} + 4X_9^{14551} - 6X_9^{14564} + 2X_9^{14575} - 2X_9^{14584} - X_9^{14635} \\
&- 6X_9^{14644} + 6X_9^{14657} - 2X_9^{14670} + 2X_9^{14679} + 3X_9^{14689} + X_9^{14691} + 4X_9^{14693} - 5X_9^{14699} \\
&+ X_9^{14715} - X_9^{14724} - X_9^{14752} - X_9^{14757} - X_9^{14768} + 2X_9^{14777} + 2X_9^{14791} + 2X_9^{14815} \\
&- 4X_9^{14824} - 6X_9^{14838} + 8X_9^{14846} - 2X_9^{14860} + 4X_9^{14869} - 2X_9^{14884} - X_9^{14910} + 2X_9^{14919} \\
&+ 8X_9^{14931} - 2X_9^{14933} - 8X_9^{14939} + 2X_9^{14955} - 4X_9^{14964} - 4X_9^{14976} + 6X_9^{14984} - X_9^{15001} \\
&+ 2X_9^{15010} + 2X_9^{15031} - X_9^{15037} + X_9^{15042} - X_9^{15044} - X_9^{15055} + 2X_9^{15102} + 2X_9^{15121} \\
&- 2X_9^{15123} - 2X_9^{15147} - X_9^{15197} + 2X_9^{15218} + 2X_9^{15242} - 2X_9^{15263} - X_9^{15288} + X_9^{15327} \\
&- X_9^{15350} - 2X_9^{15374} + 2X_9^{15397} - 2X_9^{15443} + X_9^{15457} - X_9^{15459} + X_9^{15461} + X_9^{15467} \\
&- X_9^{15492} + 2X_9^{15538} - X_9^{15585} - X_9^{15606} + X_9^{15610} - X_9^{15612} + X_9^{15614} + X_9^{15628} - 2X_9^{15649} \\
&+ 2X_9^{15651} - 2X_9^{15675} + 2X_9^{15721} - X_9^{15746} + X_9^{15770} - 2X_9^{15816} + X_9^{15863},
\end{aligned}
$$

$$
\begin{aligned}
I_{12}^{18} = \; & I_9^{18} - 2X_9^{6694} + 2X_9^{6741} - 7X_9^{6775} + 5X_9^{6781} - X_9^{6786} + X_9^{6835} + 5X_9^{6871} - 4X_9^{6877} \\
&+ 2X_9^{6882} - 3X_9^{6884} + X_9^{6916} - 2X_9^{6966} + 3X_9^{6970} - 2X_9^{6972} + 3X_9^{6974} - 3X_9^{7009} + 2X_9^{7011} \\
&+ 2X_9^{7072} - 2X_9^{7119} + 3X_9^{7158} - X_9^{7162} - 3X_9^{7164} + 3X_9^{7166} - 4X_9^{7207} + 2X_9^{7213} - 3X_9^{7218} \\
&+ 3X_9^{7220} + X_9^{7254} - 2X_9^{7258} + 4X_9^{7260} - 3X_9^{7262} + 2X_9^{7297} - 3X_9^{7299} - X_9^{7353} + X_9^{7392} \\
&+ 4X_9^{7447} - 5X_9^{7453} + 3X_9^{7458} - 2X_9^{7460} - 4X_9^{7494} + 4X_9^{7498} + 2X_9^{7500} + X_9^{7502} - 4X_9^{7537} \\
&+ 3X_9^{7539} - 3X_9^{7547} + X_9^{7588} - X_9^{7601} + X_9^{7633} - X_9^{7635} + X_9^{7637} + X_9^{7643} - X_9^{7728} + X_9^{13654} \\
&- 3X_9^{13696} - X_9^{13747} + X_9^{13783} + X_9^{13789} - 2X_9^{13794} + X_9^{13796} - 3X_9^{13830} - X_9^{13834} + 3X_9^{13836} \\
&- X_9^{13838} + X_9^{13936} - 2X_9^{13941} + 2X_9^{13983} - 2X_9^{14023} + X_9^{14029} + 3X_9^{14034} - X_9^{14036} \\
&+ 3X_9^{14070} + X_9^{14074} - 3X_9^{14076} + X_9^{14078} - 2X_9^{14121} - 2X_9^{14176} + X_9^{14181} + 3X_9^{14215} \\
&- 2X_9^{14226} + 2X_9^{14228} - 4X_9^{14262} + 5X_9^{14268} - 4X_9^{14270} - X_9^{14308} - X_9^{14321} - 2X_9^{14355} \\
&+ X_9^{14357} + X_9^{14363} + 3X_9^{14400} - X_9^{14470} + X_9^{14515} + 3X_9^{14557} + 2X_9^{14562} - 2X_9^{14564} \\
&- 2X_9^{14644} + X_9^{14657} + 2X_9^{14689} + 2X_9^{14693} - X_9^{14699} + X_9^{14752} + X_9^{14757} - 3X_9^{14797} \\
&- 2X_9^{14802} + 2X_9^{14804} + X_9^{14838} - 2X_9^{14842} - 5X_9^{14844} + 2X_9^{14846} + 2X_9^{14884} - X_9^{14897} \\
&- 2X_9^{14929} + 2X_9^{14931} - 3X_9^{14933} - 2X_9^{14976} + X_9^{14984} - X_9^{15031} + 2X_9^{15037} + X_9^{15042} \\
&- 3X_9^{15044} - X_9^{15076} + 2X_9^{15089} + 3X_9^{15121} + X_9^{15123} + X_9^{15125} + X_9^{15131} - X_9^{15327} + X_9^{15370} \\
&+ 3X_9^{15372} - 2X_9^{15374} + 2X_9^{15417} - 2X_9^{15459} + X_9^{15461} + X_9^{15467} + 2X_9^{15504} - X_9^{15512} \\
&+ X_9^{15606} - X_9^{15610} - 2X_9^{15612} + 3X_9^{15614} - X_9^{15649} + X_9^{15651} - 3X_9^{15659} - 3X_9^{15696} + X_9^{16006} \\
&- X_9^{16102} - X_9^{16144} - X_9^{16149} + X_9^{16198} + 2X_9^{16279} - X_9^{16285} + X_9^{16292} + X_9^{16384} + X_9^{16389} \\
&+ X_9^{16431} - X_9^{16480} - X_9^{16485} - 2X_9^{16519} + X_9^{16525} - X_9^{16532} - 2X_9^{16566} + X_9^{16570} + X_9^{16572} \\
&- X_9^{16574} + 2X_9^{16615} - X_9^{16621} + X_9^{16628} - X_9^{16705} + 2X_9^{16709} - X_9^{16715} - X_9^{16815} + X_9^{16911} \\
&+ 2X_9^{16950} - X_9^{16954} - X_9^{16956} + X_9^{16958} - X_9^{17001} - 2X_9^{17046} + X_9^{17050} + X_9^{17052} - X_9^{17054} \\
&+ X_9^{17089} - 2X_9^{17093} + X_9^{17099} + X_9^{17136} + X_9^{17144} - X_9^{17185} + 2X_9^{17189} - X_9^{17195} - X_9^{17279} \\
&- X_9^{17443} + X_9^{17539} - X_9^{17575} + 2X_9^{17581} + X_9^{17586} - X_9^{17635} - X_9^{17716} - X_9^{17729} + X_9^{17815} \\
&- 2X_9^{17821} - X_9^{17826} + X_9^{17862} - X_9^{17866} - 2X_9^{17868} - X_9^{17911} + 2X_9^{17917} + X_9^{17922} + X_9^{17956} \\
&+ X_9^{17969} + X_9^{18001} + X_9^{18003} - X_9^{18005} + 2X_9^{18011} - X_9^{18052} - X_9^{18065} - X_9^{18146} - X_9^{18246} \\
&+ X_9^{18250} + 2X_9^{18252} + X_9^{18297} + X_9^{18342} - X_9^{18346} - 2X_9^{18348} - X_9^{18385} - X_9^{18387} + X_9^{18389} \\
&- 2X_9^{18395} - X_9^{18432} - X_9^{18440} + X_9^{18481} + X_9^{18483} - X_9^{18485} + 2X_9^{18491} + X_9^{18530} + X_9^{18575} \\
&- X_9^{18626} - X_9^{18825} + X_9^{18921} + X_9^{18960} + X_9^{18968} - X_9^{19056} - X_9^{19064} - X_9^{19103} + X_9^{19199},
\end{aligned}
$$

$$
\begin{aligned}
I_{12}^{19} = \; & I_9^{19} - 3X_9^{21649} - 2X_9^{21655} + X_9^{21660} + X_9^{21672} - X_9^{21687} - X_9^{21734} + X_9^{21749} + X_9^{21768} \\
&+ X_9^{21782} + X_9^{21792} - 2X_9^{21797} - X_9^{21833} - X_9^{21836} - 3X_9^{21842} + 2X_9^{21847} + 2X_9^{21857}
\end{aligned}
$$

$$
\begin{aligned}
&- X_9^{21862} - X_9^{21911} + X_9^{21941} + 2X_9^{21992} + 2X_9^{21998} - 2X_9^{22018} - 2X_9^{22028} + 2X_9^{22033}\\
&- X_9^{22096} - 2X_9^{22102} + X_9^{22107} + 3X_9^{22117} - X_9^{22122} - X_9^{22132} + X_9^{22244} - X_9^{22259} - X_9^{22353}\\
&+ 3X_9^{22356} + X_9^{22362} - 2X_9^{22367} - 2X_9^{22377} + 3X_9^{22382} + X_9^{22487} - 2X_9^{22490} + 2X_9^{22496}\\
&- X_9^{22501} - X_9^{22511} - 3X_9^{22617} - X_9^{22623} + 2X_9^{22628} + X_9^{22643} + X_9^{22653} - 3X_9^{22658}\\
&+ 2X_9^{22766} - 2X_9^{22777} - 2X_9^{22787} + 2X_9^{22792} + 2X_9^{22802} + X_9^{22927} - X_9^{22942} - X_9^{23093}\\
&+ X_9^{23124} + X_9^{23252} + X_9^{23257} - X_9^{23273} - X_9^{23283} - X_9^{23419} + X_9^{23435},
\end{aligned}
$$

$$
\begin{aligned}
I_{12}^{20} = {}& I_9^{20} + X_9^{23911} - X_9^{23964} + X_9^{23999} - X_9^{24199} + X_9^{24252} - X_9^{24287} + X_9^{24487} - X_9^{24540}\\
&+ X_9^{24575} - X_9^{24739} + X_9^{24773} + X_9^{24792} - X_9^{24809} - X_9^{25189} + X_9^{25242} - X_9^{25277} + X_9^{25477}\\
&- X_9^{25530} + X_9^{25565} + X_9^{25603} - X_9^{25637} - X_9^{25656} + X_9^{25673} - X_9^{25765} + X_9^{25818} - X_9^{25853}\\
&- X_9^{26323} + X_9^{26357} + X_9^{26376} - X_9^{26393} + X_9^{26611} - X_9^{26645} - X_9^{26664} + X_9^{26681},
\end{aligned}
$$

$$
\begin{aligned}
I_{12}^{21} = {}& I_9^{21} + X_9^{23890} - X_9^{23905} - 3X_9^{23911} + 2X_9^{23917} - X_9^{23922} - X_9^{23943} - X_9^{23953} + 2X_9^{23958}\\
&+ 3X_9^{23964} - X_9^{23970} - X_9^{23992} - 3X_9^{23999} + X_9^{24004} - X_9^{24178} + X_9^{24193} + 3X_9^{24199}\\
&- 2X_9^{24205} + X_9^{24210} + X_9^{24231} + X_9^{24241} - 2X_9^{24246} - 3X_9^{24252} + X_9^{24258} + X_9^{24280}\\
&+ 3X_9^{24287} - X_9^{24292} + X_9^{24466} - X_9^{24481} - 3X_9^{24487} + 2X_9^{24493} - X_9^{24498} - X_9^{24519} - X_9^{24529}\\
&+ 2X_9^{24534} + 3X_9^{24540} - X_9^{24546} - X_9^{24568} - 3X_9^{24575} + X_9^{24580} + 2X_9^{24739} + X_9^{24745}\\
&- X_9^{24750} + X_9^{24766} - 2X_9^{24778} + X_9^{24783} - X_9^{24792} - X_9^{24798} + X_9^{24803} + X_9^{24814} - 2X_9^{24819}\\
&+ X_9^{24831} + X_9^{24854} - X_9^{24866} - X_9^{25185} + 2X_9^{25189} + X_9^{25200} + 2X_9^{25206} - 2X_9^{25212} + X_9^{25217}\\
&+ X_9^{25238} - 4X_9^{25242} + X_9^{25248} - 2X_9^{25253} + X_9^{25265} + 3X_9^{25277} + X_9^{25287} - X_9^{25299} + X_9^{25473}\\
&- 2X_9^{25477} - X_9^{25488} - 2X_9^{25494} + 2X_9^{25500} - X_9^{25505} - X_9^{25526} + 4X_9^{25530} - X_9^{25536}\\
&+ 2X_9^{25541} - X_9^{25553} - 3X_9^{25565} - X_9^{25575} + X_9^{25587} - 2X_9^{25603} - X_9^{25609} + X_9^{25614}\\
&- X_9^{25630} + 2X_9^{25642} - X_9^{25647} + X_9^{25656} + X_9^{25662} - X_9^{25667} - X_9^{25678} + 2X_9^{25683} - X_9^{25695}\\
&- X_9^{25718} + X_9^{25730} - X_9^{25761} + 2X_9^{25765} + X_9^{25776} + 2X_9^{25782} - 2X_9^{25788} + X_9^{25793} + X_9^{25814}\\
&- 4X_9^{25818} + X_9^{25824} - 2X_9^{25829} + X_9^{25841} + 3X_9^{25853} + X_9^{25863} - X_9^{25875} + 2X_9^{26323}\\
&+ X_9^{26329} - X_9^{26334} + X_9^{26350} - 2X_9^{26362} + X_9^{26367} - X_9^{26376} - X_9^{26382} + X_9^{26387} + X_9^{26398}\\
&- 2X_9^{26403} + X_9^{26415} + X_9^{26438} - X_9^{26450} - 2X_9^{26611} - X_9^{26617} + X_9^{26622} - X_9^{26638} + 2X_9^{26650}\\
&- X_9^{26655} + X_9^{26664} + X_9^{26670} - X_9^{26675} - X_9^{26686} + 2X_9^{26691} - X_9^{26703} - X_9^{26726} + X_9^{26738},
\end{aligned}
$$

$$
\begin{aligned}
I_{12}^{22} = {}& I_9^{22} + X_9^{23839} - X_9^{23856} + X_9^{23911} - X_9^{23947} + X_9^{23999} - X_9^{24235} + X_9^{24252} + X_9^{24361}\\
&- X_9^{24395} + X_9^{24487} - X_9^{24523} + X_9^{24575} + X_9^{24613} - 2X_9^{24630} + X_9^{24666} - X_9^{24701} - X_9^{24739}\\
&+ X_9^{24773} - X_9^{24991} + X_9^{25008} - X_9^{25206} + X_9^{25242} - X_9^{25277} - X_9^{25351} + 2X_9^{25387} - X_9^{25404}\\
&- X_9^{25439} + X_9^{25494} - X_9^{25530} + X_9^{25565} - X_9^{25656} + X_9^{25673} - X_9^{25765} + X_9^{25782} + X_9^{26034}\\
&- X_9^{26051} - X_9^{26233} + X_9^{26267} + X_9^{26376} - X_9^{26393} + X_9^{26611} - X_9^{26645} - X_9^{26754} + X_9^{26771},
\end{aligned}
$$

$$
\begin{aligned}
I_{12}^{23} = {}& I_9^{23} + X_9^{23764} - X_9^{23781} - X_9^{23817} + X_9^{23834} + X_9^{23839} - X_9^{23845} + X_9^{23850} - 2X_9^{23856}\\
&+ X_9^{23862} - X_9^{23867} - X_9^{23905} - X_9^{23911} + X_9^{23922} - X_9^{23947} + X_9^{23958} + X_9^{23964} - X_9^{23975}\\
&- X_9^{23992} + X_9^{24009} - X_9^{24178} + 2X_9^{24199} - 2X_9^{24205} + 2X_9^{24210} + X_9^{24231} - X_9^{24235}\\
&+ X_9^{24241} - X_9^{24246} - 2X_9^{24252} + X_9^{24258} - X_9^{24263} + 3X_9^{24287} - X_9^{24292} + X_9^{24297} + X_9^{24319}\\
&- 2X_9^{24352} + X_9^{24361} - X_9^{24372} + X_9^{24388} + X_9^{24405} - X_9^{24440} - X_9^{24481} - X_9^{24487} + X_9^{24498}\\
&- X_9^{24523} + X_9^{24534} + X_9^{24540} - X_9^{24551} - X_9^{24568} + X_9^{24585} + X_9^{24609} + 2X_9^{24619} - X_9^{24624}\\
&- 2X_9^{24630} - X_9^{24641} - X_9^{24662} + 4X_9^{24666} - 3X_9^{24672} + 2X_9^{24677} + X_9^{24689} - 3X_9^{24701}\\
&+ 2X_9^{24706} - X_9^{24711} - X_9^{24723} + X_9^{24739} - X_9^{24750} + X_9^{24766} + X_9^{24783} - X_9^{24792} + X_9^{24803}\\
&- 2X_9^{24819} + X_9^{24854} - X_9^{24916} + X_9^{24933} + X_9^{24969} - X_9^{24986} - X_9^{24991} + X_9^{24997} - X_9^{25002}\\
&+ 2X_9^{25008} - X_9^{25014} + X_9^{25019} + X_9^{25189} + X_9^{25195} - X_9^{25212} - X_9^{25248} + X_9^{25265} + X_9^{25282}\\
&- X_9^{25299} + X_9^{25330} + X_9^{25345} - X_9^{25351} + 2X_9^{25357} - 3X_9^{25362} - X_9^{25383} + 2X_9^{25387}
\end{aligned}
$$

$$
\begin{aligned}
&- X_9^{25393} + X_9^{25404} - X_9^{25410} + 2X_9^{25415} + X_9^{25432} - 3X_9^{25439} + X_9^{25444} - 2X_9^{25449} \\
&- X_9^{25477} - X_9^{25483} + X_9^{25500} + X_9^{25536} - X_9^{25553} - X_9^{25570} + X_9^{25587} - X_9^{25603} - X_9^{25609} \\
&+ 2X_9^{25642} + X_9^{25662} - X_9^{25678} - X_9^{25695} + X_9^{25730} - X_9^{25761} + X_9^{25765} - X_9^{25771} + X_9^{25776} \\
&+ 2X_9^{25782} - X_9^{25788} + X_9^{25793} + X_9^{25814} - 4X_9^{25818} + 2X_9^{25824} - 2X_9^{25829} + 3X_9^{25853} \\
&- X_9^{25858} + X_9^{25863} + X_9^{26040} - X_9^{26056} - X_9^{26073} - X_9^{26093} + 2X_9^{26109} - X_9^{26144} - X_9^{26191} \\
&+ 2X_9^{26224} - X_9^{26233} + X_9^{26244} - X_9^{26260} - X_9^{26277} + X_9^{26312} + X_9^{26323} + X_9^{26329} - 2X_9^{26362} \\
&- X_9^{26382} + X_9^{26398} + X_9^{26415} - X_9^{26450} - X_9^{26611} + X_9^{26622} - X_9^{26638} - X_9^{26655} + X_9^{26664} \\
&- X_9^{26675} + 2X_9^{26691} - X_9^{26726} - X_9^{26760} + X_9^{26776} + X_9^{26793} + X_9^{26813} - 2X_9^{26829} + X_9^{26864},
\end{aligned}
$$

$$
\begin{aligned}
I_{12}^{24} = {} & I_9^{24} + X_9^{23713} - X_9^{23856} + X_9^{23999} - X_9^{24109} + X_9^{24252} - X_9^{24395} + 2X_9^{24487} - X_9^{24523} \\
&- 2X_9^{24630} + X_9^{24666} + 2X_9^{24773} - X_9^{24809} - X_9^{24991} + X_9^{25134} - X_9^{25277} - X_9^{25351} \\
&+ 2X_9^{25387} + X_9^{25494} - 2X_9^{25530} - X_9^{25637} + 2X_9^{25673} - X_9^{25765} + X_9^{25908} - X_9^{26051} \\
&- X_9^{26233} + X_9^{26376} - X_9^{26519} + X_9^{26611} - X_9^{26754} + X_9^{26897},
\end{aligned}
$$

$$
\begin{aligned}
I_{12}^{25} = {} & I_9^{25} - X_9^{23638} + X_9^{23674} - X_9^{23707} - 2X_9^{23713} + X_9^{23719} + X_9^{23781} - X_9^{23817} - X_9^{23839} \\
&+ X_9^{23850} + 3X_9^{23856} - X_9^{23862} - X_9^{23911} + X_9^{23917} - X_9^{23922} + X_9^{23947} - X_9^{23953} + X_9^{23958} \\
&- X_9^{23992} - 3X_9^{23999} + X_9^{24004} - X_9^{24088} + X_9^{24103} + 2X_9^{24109} - X_9^{24115} + X_9^{24231} + X_9^{24235} \\
&- X_9^{24246} - 3X_9^{24252} + X_9^{24258} - X_9^{24361} + X_9^{24367} - X_9^{24372} + X_9^{24388} + 3X_9^{24395} - X_9^{24400} \\
&+ X_9^{24466} - X_9^{24481} - 4X_9^{24487} + X_9^{24498} + 2X_9^{24523} + X_9^{24529} - X_9^{24534} + X_9^{24568} - X_9^{24580} \\
&- X_9^{24609} - X_9^{24613} + X_9^{24624} + 4X_9^{24630} - X_9^{24641} - X_9^{24666} - X_9^{24672} + X_9^{24677} - X_9^{24711} \\
&+ X_9^{24723} + X_9^{24739} - X_9^{24745} + X_9^{24750} - X_9^{24766} - 3X_9^{24773} + X_9^{24783} + X_9^{24814} - X_9^{24819} \\
&+ X_9^{24854} - X_9^{24866} + X_9^{24933} - X_9^{24969} + 2X_9^{24991} + X_9^{25002} - X_9^{25014} - X_9^{25076} + X_9^{25112} \\
&- 2X_9^{25134} - X_9^{25145} + X_9^{25157} + 2X_9^{25206} - X_9^{25212} + X_9^{25217} - 2X_9^{25242} + X_9^{25248} \\
&- X_9^{25253} + 3X_9^{25277} + X_9^{25287} - X_9^{25299} + 2X_9^{25351} + X_9^{25357} - X_9^{25362} + X_9^{25383} - 4X_9^{25387} \\
&- X_9^{25393} + X_9^{25410} - X_9^{25432} + X_9^{25444} - X_9^{25494} - X_9^{25500} + X_9^{25505} - X_9^{25526} + 3X_9^{25530} \\
&+ X_9^{25536} - X_9^{25553} + X_9^{25575} - X_9^{25587} + X_9^{25642} - X_9^{25647} + 2X_9^{25656} - X_9^{25662} + X_9^{25667} \\
&- 3X_9^{25673} - X_9^{25678} + X_9^{25695} - X_9^{25718} + X_9^{25730} - X_9^{25761} + 2X_9^{25765} + X_9^{25776} - X_9^{25788} \\
&+ X_9^{25904} - 2X_9^{25908} - X_9^{25919} + X_9^{25931} - 2X_9^{26034} + X_9^{26040} - X_9^{26045} + 3X_9^{26051} \\
&+ X_9^{26061} - X_9^{26073} + 2X_9^{26233} + X_9^{26239} - X_9^{26244} + X_9^{26260} - X_9^{26272} - X_9^{26376} - X_9^{26382} \\
&+ X_9^{26387} - X_9^{26403} + X_9^{26415} + X_9^{26524} - X_9^{26529} + X_9^{26546} - X_9^{26558} - 2X_9^{26611} - X_9^{26617} \\
&+ X_9^{26622} - X_9^{26638} + X_9^{26650} + X_9^{26754} + X_9^{26760} - X_9^{26765} + X_9^{26781} - X_9^{26793} - X_9^{26902} \\
&+ X_9^{26907} - X_9^{26924} + X_9^{26936},
\end{aligned}
$$

$$
\begin{aligned}
I_{12}^{26} = {} & I_9^{26} + X_9^{23674} - X_9^{23691} - X_9^{23707} - X_9^{23713} + X_9^{23724} - X_9^{23817} + X_9^{23834} - X_9^{23839} \\
&+ X_9^{23850} + X_9^{23856} - X_9^{23867} + X_9^{23947} - X_9^{23953} + X_9^{23958} - 2X_9^{23964} + X_9^{23970} - X_9^{23975} \\
&- X_9^{23992} + X_9^{24009} - X_9^{24067} - X_9^{24088} + X_9^{24103} + X_9^{24109} - X_9^{24115} + X_9^{24120} - X_9^{24154} \\
&- X_9^{24199} + X_9^{24210} + X_9^{24231} + X_9^{24235} - X_9^{24246} - 2X_9^{24252} + X_9^{24258} - X_9^{24263} + X_9^{24297} \\
&- X_9^{24352} - X_9^{24361} + X_9^{24367} - X_9^{24372} + X_9^{24388} + 3X_9^{24395} - X_9^{24400} + X_9^{24405} - X_9^{24440} \\
&- X_9^{24481} - 2X_9^{24487} + 2X_9^{24498} + X_9^{24519} + X_9^{24523} + 2X_9^{24529} - 2X_9^{24534} - X_9^{24546} + X_9^{24568} \\
&- X_9^{24613} + X_9^{24624} + 2X_9^{24630} - 2X_9^{24641} - X_9^{24662} + X_9^{24666} - 2X_9^{24672} + 2X_9^{24677} + X_9^{24689} \\
&- X_9^{24711} - X_9^{24766} + 2X_9^{24783} + 2X_9^{24792} - X_9^{24798} + X_9^{24803} - 3X_9^{24809} + 2X_9^{24814} \\
&- 2X_9^{24819} - X_9^{24831} + X_9^{24854} - X_9^{24916} + X_9^{24933} + X_9^{24991} + X_9^{24997} - X_9^{25014} + X_9^{25059} \\
&- X_9^{25076} - X_9^{25140} + X_9^{25157} - X_9^{25189} + X_9^{25195} - X_9^{25200} + 2X_9^{25206} - X_9^{25212} + X_9^{25217} \\
&+ X_9^{25282} - X_9^{25299} + X_9^{25330} + X_9^{25345} + X_9^{25351} + 2X_9^{25357} - 2X_9^{25362} - 2X_9^{25387} - 2X_9^{25393} \\
&+ X_9^{25410} + X_9^{25444} - X_9^{25473} + X_9^{25477} - X_9^{25488} + X_9^{25494} - 2X_9^{25500} + 2X_9^{25505} + 2X_9^{25536} \\
&- X_9^{25553} - X_9^{25587} + X_9^{25603} - X_9^{25609} + X_9^{25614} + X_9^{25630} - 3X_9^{25637} + 2X_9^{25642} - 2X_9^{25647}
\end{aligned}
$$

$$
\begin{aligned}
&-2X_9^{25678}+X_9^{25695}+X_9^{25730}-X_9^{25761}+X_9^{25765}-X_9^{25771}+X_9^{25776}-X_9^{25788}+X_9^{25824}\\
&-X_9^{25858}+X_9^{25904}-2X_9^{25908}+X_9^{25914}-X_9^{25919}+X_9^{25931}-X_9^{25967}+X_9^{26001}-2X_9^{26034}\\
&+X_9^{26040}-X_9^{26045}+3X_9^{26051}-X_9^{26056}+X_9^{26061}-X_9^{26073}+X_9^{26109}-X_9^{26144}-X_9^{26191}\\
&+X_9^{26224}+X_9^{26233}+X_9^{26239}-X_9^{26272}-X_9^{26323}+X_9^{26334}-X_9^{26367}-X_9^{26382}+X_9^{26415}\\
&-X_9^{26476}+X_9^{26510}+X_9^{26524}-X_9^{26558}-X_9^{26611}+X_9^{26622}-X_9^{26638}-X_9^{26670}+X_9^{26686}\\
&+X_9^{26754}-X_9^{26765}+X_9^{26781}+X_9^{26813}-X_9^{26829}+X_9^{26907}-X_9^{26924}-X_9^{26955}+X_9^{26972},
\end{aligned}
$$

$$
\begin{aligned}
I_{12}^{27} = {} & I_9^{27}-X_9^{23638}+2X_9^{23674}-X_9^{23691}-X_9^{23713}+X_9^{23719}-X_9^{23724}-2X_9^{23764}+3X_9^{23781}\\
&-X_9^{23834}-2X_9^{23839}-4X_9^{23845}+4X_9^{23850}+3X_9^{23862}-3X_9^{23867}+2X_9^{23890}-X_9^{23911}\\
&+2X_9^{23917}-3X_9^{23922}-X_9^{23943}+2X_9^{23947}+2X_9^{23953}+X_9^{23964}-3X_9^{23970}+2X_9^{23975}\\
&-3X_9^{23992}+2X_9^{24009}-X_9^{24088}+X_9^{24109}-X_9^{24115}+X_9^{24120}+2X_9^{24193}-X_9^{24199}\\
&-3X_9^{24205}+X_9^{24210}+X_9^{24231}+2X_9^{24235}+6X_9^{24241}-8X_9^{24246}+2X_9^{24252}-2X_9^{24258}\\
&+4X_9^{24263}-X_9^{24280}+3X_9^{24287}-3X_9^{24292}+4X_9^{24297}-2X_9^{24319}-2X_9^{24361}-3X_9^{24367}\\
&+2X_9^{24372}+6X_9^{24388}-3X_9^{24395}+3X_9^{24400}-6X_9^{24405}-2X_9^{24440}+2X_9^{24466}-X_9^{24481}\\
&-3X_9^{24487}+2X_9^{24493}-X_9^{24498}-X_9^{24519}+3X_9^{24523}+2X_9^{24529}-X_9^{24534}+X_9^{24540}\\
&-3X_9^{24546}+2X_9^{24551}-X_9^{24568}+X_9^{24585}-2X_9^{24609}-X_9^{24613}-7X_9^{24619}+6X_9^{24624}\\
&-2X_9^{24630}+8X_9^{24636}-7X_9^{24641}+X_9^{24662}-2X_9^{24666}-X_9^{24672}+2X_9^{24677}-X_9^{24689}\\
&-3X_9^{24701}+5X_9^{24706}-3X_9^{24711}-2X_9^{24723}-X_9^{24739}+4X_9^{24745}-X_9^{24750}-3X_9^{24766}\\
&+6X_9^{24773}-4X_9^{24778}+7X_9^{24783}+X_9^{24792}+X_9^{24798}-X_9^{24803}-4X_9^{24814}-2X_9^{24819}\\
&+3X_9^{24831}+3X_9^{24854}+X_9^{24916}-3X_9^{24969}+2X_9^{24986}+3X_9^{24991}+3X_9^{24997}-3X_9^{25002}\\
&+2X_9^{25008}-4X_9^{25014}+4X_9^{25019}+X_9^{25059}-2X_9^{25076}+X_9^{25112}-2X_9^{25134}+X_9^{25140}\\
&-X_9^{25145}-2X_9^{25185}-2X_9^{25189}-X_9^{25195}+X_9^{25200}+6X_9^{25206}-2X_9^{25212}+3X_9^{25217}\\
&+X_9^{25238}-3X_9^{25242}+X_9^{25248}-3X_9^{25253}+X_9^{25265}+3X_9^{25277}+2X_9^{25287}-X_9^{25299}\\
&-X_9^{25330}-X_9^{25345}+3X_9^{25351}+2X_9^{25357}-X_9^{25362}+2X_9^{25383}-6X_9^{25387}-7X_9^{25393}\\
&+8X_9^{25398}-5X_9^{25404}+6X_9^{25410}-7X_9^{25415}+2X_9^{25432}-3X_9^{25439}+3X_9^{25444}-5X_9^{25449}\\
&+X_9^{25473}+2X_9^{25477}+2X_9^{25483}-3X_9^{25488}-2X_9^{25494}-X_9^{25500}+2X_9^{25505}-2X_9^{25526}\\
&+4X_9^{25530}-X_9^{25536}+2X_9^{25541}-X_9^{25553}-X_9^{25570}+X_9^{25587}+2X_9^{25603}-X_9^{25609}\\
&+3X_9^{25614}-X_9^{25630}-6X_9^{25637}+2X_9^{25642}-4X_9^{25647}+X_9^{25662}-X_9^{25678}+2X_9^{25683}\\
&-X_9^{25695}-2X_9^{25718}+X_9^{25730}-X_9^{25761}+4X_9^{25771}-2X_9^{25776}+6X_9^{25782}-8X_9^{25788}\\
&+6X_9^{25793}+X_9^{25824}-3X_9^{25829}+2X_9^{25841}+3X_9^{25853}-4X_9^{25858}+3X_9^{25863}+X_9^{25875}\\
&+X_9^{25904}-2X_9^{25908}+X_9^{25914}-X_9^{25919}-2X_9^{26034}-2X_9^{26040}-X_9^{26045}+3X_9^{26051}\\
&+2X_9^{26056}+X_9^{26061}+X_9^{26191}+3X_9^{26233}+3X_9^{26239}-3X_9^{26244}-3X_9^{26260}+3X_9^{26267}\\
&-3X_9^{26272}+5X_9^{26277}+X_9^{26312}-2X_9^{26323}-X_9^{26334}+2X_9^{26350}+3X_9^{26357}-X_9^{26362}\\
&+X_9^{26376}-2X_9^{26382}+2X_9^{26387}-3X_9^{26393}+2X_9^{26398}-3X_9^{26403}+X_9^{26415}+X_9^{26438}\\
&-X_9^{26450}-X_9^{26476}+X_9^{26510}-X_9^{26529}+X_9^{26546}-4X_9^{26617}+2X_9^{26622}+X_9^{26638}-6X_9^{26645}\\
&+4X_9^{26650}-5X_9^{26655}-X_9^{26664}-X_9^{26670}+X_9^{26675}+4X_9^{26686}+X_9^{26691}-3X_9^{26703}\\
&-2X_9^{26726}+X_9^{26754}+3X_9^{26760}-X_9^{26765}-3X_9^{26776}+X_9^{26781}+X_9^{26907}-X_9^{26924}-X_9^{27166}\\
&+X_9^{27183}-X_9^{27238}+2X_9^{27274}-X_9^{27291}+X_9^{27307}+X_9^{27319}-2X_9^{27324}-X_9^{27364}+2X_9^{27381}\\
&-X_9^{27417}+X_9^{27439}-3X_9^{27445}+2X_9^{27450}-2X_9^{27456}+2X_9^{27462}-X_9^{27467}+X_9^{27523}\\
&-X_9^{27544}-2X_9^{27559}-X_9^{27571}+2X_9^{27576}+X_9^{27610}+X_9^{27634}+X_9^{27649}+X_9^{27655}-X_9^{27661}\\
&-2X_9^{27666}+X_9^{27687}-2X_9^{27691}+5X_9^{27697}-2X_9^{27702}+2X_9^{27708}-2X_9^{27714}+X_9^{27719}\\
&-2X_9^{27736}+3X_9^{27743}-2X_9^{27748}+2X_9^{27753}-X_9^{27777}+X_9^{27781}-3X_9^{27787}+2X_9^{27792}\\
&-4X_9^{27798}+4X_9^{27804}-2X_9^{27809}+2X_9^{27834}-2X_9^{27840}+X_9^{27845}-3X_9^{27869}+3X_9^{27874}\\
&-X_9^{27879}-X_9^{27891}+X_9^{27973}-X_9^{28006}-X_9^{28063}+X_9^{28105}-2X_9^{28111}+3X_9^{28132}-3X_9^{28139}\\
&+2X_9^{28144}-2X_9^{28149}-X_9^{28184}-X_9^{28195}+2X_9^{28201}-2X_9^{28222}+3X_9^{28229}-X_9^{28234}
\end{aligned}
$$

$$
\begin{aligned}
&+3X_9^{28239}-2X_9^{28248}+2X_9^{28254}-X_9^{28259}+3X_9^{28265}-4X_9^{28270}+X_9^{28287}+X_9^{28310}\\
&+X_9^{28322}+2X_9^{28338}-2X_9^{28344}+X_9^{28349}-3X_9^{28355}+3X_9^{28360}-X_9^{28365}-2X_9^{28377}\\
&+X_9^{28413}-X_9^{28448}+X_9^{28605}-X_9^{28622}+X_9^{28677}-2X_9^{28713}+X_9^{28730}+X_9^{28741}-2X_9^{28746}\\
&+3X_9^{28752}-2X_9^{28758}+3X_9^{28763}+X_9^{28803}-2X_9^{28820}+X_9^{28856}-2X_9^{28878}+2X_9^{28884}\\
&-X_9^{28889}-X_9^{28901}+X_9^{28957}-2X_9^{28962}+X_9^{28983}-2X_9^{28993}+4X_9^{28998}-3X_9^{29004}\\
&+2X_9^{29010}-3X_9^{29015}+X_9^{29032}-3X_9^{29039}+X_9^{29044}-3X_9^{29049}-X_9^{29073}+X_9^{29083}\\
&-2X_9^{29088}+4X_9^{29094}-2X_9^{29100}+5X_9^{29105}-X_9^{29126}+X_9^{29130}-2X_9^{29136}-X_9^{29141}\\
&+X_9^{29153}+3X_9^{29165}-2X_9^{29170}+2X_9^{29175}+2X_9^{29187}+X_9^{29216}-2X_9^{29220}+2X_9^{29226}\\
&-X_9^{29231}-2X_9^{29243}+X_9^{29279}-X_9^{29313}+X_9^{29392}+X_9^{29407}-2X_9^{29412}-2X_9^{29428}+3X_9^{29435}\\
&-X_9^{29440}+3X_9^{29445}+X_9^{29480}-X_9^{29497}+2X_9^{29502}+X_9^{29518}-3X_9^{29525}-4X_9^{29535}\\
&+X_9^{29544}+2X_9^{29555}-3X_9^{29561}+3X_9^{29566}-X_9^{29571}-2X_9^{29583}-X_9^{29606}-X_9^{29618}\\
&-X_9^{29634}-2X_9^{29645}+3X_9^{29651}-2X_9^{29656}+2X_9^{29661}+3X_9^{29673}-X_9^{29693}+X_9^{29744}\\
&+X_9^{29783}-X_9^{29799}+X_9^{29986}-X_9^{30020}-X_9^{30076}+X_9^{30110}-X_9^{30124}-X_9^{30129}+X_9^{30146}\\
&+X_9^{30158}+X_9^{30214}+X_9^{30219}-X_9^{30236}-X_9^{30248}+X_9^{30267}-X_9^{30284}-X_9^{30357}+X_9^{30374},
\end{aligned}
$$

$$
\begin{aligned}
I_{12}^{28}=\;&I_9^{28}+3X_9^{21655}+2X_9^{21666}-4X_9^{21672}+X_9^{21677}+X_9^{21687}-X_9^{21692}+X_9^{30514}-2X_9^{30531}\\
&+X_9^{30535}-3X_9^{30541}+2X_9^{30546}+X_9^{30566}-4X_9^{30570}+4X_9^{30576}-2X_9^{30581}+3X_9^{30587}\\
&-3X_9^{30592}+X_9^{30597}-X_9^{30619}-X_9^{30625}+X_9^{30631}+2X_9^{30636}-2X_9^{30652}+X_9^{30660}\\
&-2X_9^{30666}-X_9^{30671}+2X_9^{30682}+2X_9^{30687}-X_9^{30704}+X_9^{30719}-2X_9^{30735}+X_9^{30752},
\end{aligned}
$$

$$
\begin{aligned}
I_{12}^{29}=\;&I_9^{29}-X_9^{21649}+X_9^{21655}-X_9^{21660}-X_9^{21672}+X_9^{21677}-X_9^{21692}-X_9^{21786}+X_9^{21836}\\
&-X_9^{22096}+X_9^{22356}-X_9^{22490}-X_9^{22617}+2X_9^{22766}-X_9^{23247}-X_9^{30535}+2X_9^{30570}+X_9^{30587}\\
&+X_9^{30625}-X_9^{30660}-2X_9^{30677}+X_9^{30859}-2X_9^{30949}+2X_9^{31021}-X_9^{31073}+X_9^{31147}\\
&-2X_9^{31219}+2X_9^{31271}+X_9^{31309}-2X_9^{31343}-X_9^{31506}+2X_9^{31596}+X_9^{31613}-2X_9^{31668}\\
&-X_9^{31794}-2X_9^{31811}+2X_9^{31866}+2X_9^{31883}-X_9^{31956}+X_9^{32117}-2X_9^{32189}+X_9^{32279},
\end{aligned}
$$

$$
\begin{aligned}
I_{12}^{30}=\;&I_9^{30}+2X_9^{21655}-4X_9^{21660}+X_9^{21666}-X_9^{21672}+2X_9^{21677}-2X_9^{21687}+X_9^{21734}-X_9^{21749}\\
&-X_9^{21768}-X_9^{21782}-X_9^{21786}+X_9^{21797}+2X_9^{21836}+2X_9^{21842}-X_9^{21847}-2X_9^{21857}\\
&+X_9^{21862}+X_9^{21911}-2X_9^{21998}+X_9^{22003}+X_9^{22018}-X_9^{22033}-X_9^{22117}+X_9^{22132}-X_9^{22244}\\
&+X_9^{22259}+X_9^{22353}-X_9^{22356}-X_9^{22362}+2X_9^{22367}+X_9^{22377}-2X_9^{22382}-X_9^{22496}+X_9^{22511}\\
&+X_9^{22623}-2X_9^{22628}-X_9^{22643}+X_9^{22658}+X_9^{22772}+X_9^{22787}-X_9^{22802}+X_9^{23093}-X_9^{23252}\\
&-X_9^{30514}+2X_9^{30531}-X_9^{30535}+X_9^{30546}-X_9^{30566}+X_9^{30570}-X_9^{30581}+3X_9^{30587}-X_9^{30592}\\
&+2X_9^{30597}+X_9^{30619}+X_9^{30625}+X_9^{30631}-3X_9^{30636}+X_9^{30660}-X_9^{30666}+2X_9^{30671}\\
&-3X_9^{30677}+X_9^{30682}-X_9^{30687}-X_9^{30704}+X_9^{30820}-X_9^{30855}+X_9^{30859}-X_9^{30865}+X_9^{30870}\\
&-X_9^{30910}-X_9^{30925}+X_9^{30945}-2X_9^{30949}+X_9^{30955}-X_9^{30976}+X_9^{31000}-X_9^{31017}+X_9^{31021}\\
&+X_9^{31027}+2X_9^{31056}-2X_9^{31062}+X_9^{31067}-3X_9^{31073}+2X_9^{31078}-X_9^{31083}+X_9^{31123}\\
&+X_9^{31147}-X_9^{31158}+X_9^{31174}-X_9^{31213}-X_9^{31219}-2X_9^{31225}+2X_9^{31230}-2X_9^{31254}\\
&+2X_9^{31260}-X_9^{31265}+3X_9^{31271}-2X_9^{31276}+X_9^{31298}+2X_9^{31315}-X_9^{31320}+2X_9^{31326}\\
&-2X_9^{31332}+X_9^{31337}-3X_9^{31343}+X_9^{31348}-X_9^{31353}+X_9^{31383}-X_9^{31400}-X_9^{31467}+X_9^{31502}\\
&-2X_9^{31506}+X_9^{31512}-X_9^{31517}+X_9^{31557}-X_9^{31567}+2X_9^{31572}-X_9^{31592}+X_9^{31596}-X_9^{31607}\\
&+3X_9^{31613}-X_9^{31618}+2X_9^{31623}-X_9^{31647}+X_9^{31664}-2X_9^{31668}-X_9^{31679}+X_9^{31709}\\
&-X_9^{31725}+X_9^{31765}-2X_9^{31770}-X_9^{31786}+X_9^{31794}-X_9^{31800}+2X_9^{31805}-3X_9^{31811}+X_9^{31816}\\
&-X_9^{31821}-X_9^{31838}-X_9^{31855}+2X_9^{31860}-X_9^{31866}+2X_9^{31872}-2X_9^{31877}+3X_9^{31883}\\
&+2X_9^{31893}-X_9^{31907}+X_9^{31940}-X_9^{31962}+X_9^{31979}-X_9^{31995}+X_9^{32092}-X_9^{32127}+X_9^{32144}\\
&-X_9^{32182}-X_9^{32194}+X_9^{32199}-X_9^{32216}+X_9^{32229}-X_9^{32246}+X_9^{32284}-X_9^{32301}+X_9^{32318},
\end{aligned}
$$

$$
\begin{aligned}
I_{12}^{31} = {}& I_9^{31} + X_9^{399} - X_9^{402} - X_9^{426} + X_9^{429} + X_9^{444} + X_9^{454} - X_9^{457} + X_9^{478} - X_9^{508} - X_9^{534} \\
& - X_9^{538} + X_9^{539} + X_9^{564} - X_9^{597} - X_9^{36070} + X_9^{36072} + X_9^{36091} - X_9^{36096} - X_9^{36125} + 2X_9^{36130} \\
& + X_9^{36137} - X_9^{36139} + X_9^{36167} + X_9^{36174} - X_9^{36176} - X_9^{36192} - X_9^{36200} - X_9^{36209} + X_9^{36211} \\
& + X_9^{36226} - X_9^{36237} + X_9^{36263} + X_9^{36266} - X_9^{36268} - X_9^{36297} + X_9^{36303},
\end{aligned}
$$

$$
\begin{aligned}
I_{12}^{32} = {}& I_9^{32} - X_9^{399} + X_9^{402} + X_9^{426} - X_9^{429} + X_9^{444} - X_9^{447} + X_9^{454} - X_9^{457} - X_9^{478} - X_9^{505} \\
& + X_9^{508} - X_9^{515} + X_9^{518} - X_9^{534} - X_9^{538} + X_9^{539} + X_9^{724} - X_9^{727} - X_9^{801} + X_9^{804} + X_9^{833} \\
& + X_9^{882} - X_9^{885} - X_9^{955} - X_9^{965} + X_9^{968} - X_9^{1048} + X_9^{1141} + X_9^{1185} + X_9^{1195} - X_9^{1198} \\
& + X_9^{1231} + X_9^{1235} - X_9^{1236} - X_9^{1284} - X_9^{1294} + X_9^{1297} + X_9^{1380} - X_9^{1560} + X_9^{1563} + X_9^{1673} \\
& - X_9^{1676} + X_9^{1720} - X_9^{1723} + X_9^{1730} - X_9^{1733} - X_9^{1790} + X_9^{1793} + X_9^{1894} - X_9^{2025} + X_9^{2028} \\
& - X_9^{2035} + X_9^{2038} - X_9^{2086} - X_9^{2090} + X_9^{2091} + X_9^{2154} - X_9^{2157} + X_9^{2164} - X_9^{2167} - X_9^{2214} \\
& - X_9^{2280} + X_9^{2349} + X_9^{2634} + X_9^{2638} - X_9^{2639} - X_9^{2781} - X_9^{2785} + X_9^{2786} + X_9^{2852} - X_9^{3005} \\
& + X_9^{36070} - X_9^{36072} - X_9^{36091} + X_9^{36096} - X_9^{36120} - X_9^{36125} - X_9^{36128} + 2X_9^{36130} - X_9^{36167} \\
& - X_9^{36174} + X_9^{36176} + X_9^{36192} + X_9^{36197} + X_9^{36200} - 2X_9^{36202} + X_9^{36226} - X_9^{36237} - X_9^{36430} \\
& + X_9^{36432} + X_9^{36502} - X_9^{36504} + X_9^{36527} + X_9^{36534} - X_9^{36536} - X_9^{36574} + X_9^{36576} + X_9^{36629} \\
& - 2X_9^{36634} - X_9^{36641} + X_9^{36643} - X_9^{36707} - X_9^{36714} + X_9^{36716} + X_9^{36779} + X_9^{36786} - X_9^{36788} \\
& - X_9^{36809} + 2X_9^{36814} + X_9^{36821} - X_9^{36823} - X_9^{36838} + X_9^{36849} + X_9^{36881} - 2X_9^{36886} - X_9^{36893} \\
& + X_9^{36895} + X_9^{36945} - X_9^{36951} + X_9^{37063} - X_9^{37068} - X_9^{37135} + X_9^{37140} - X_9^{37164} - X_9^{37169} \\
& - X_9^{37172} + 2X_9^{37174} + X_9^{37207} - X_9^{37212} + X_9^{37271} + X_9^{37274} - X_9^{37276} + X_9^{37344} + X_9^{37349} \\
& + X_9^{37352} - 2X_9^{37354} + X_9^{37378} - X_9^{37389} - X_9^{37416} - X_9^{37421} - X_9^{37424} + 2X_9^{37426} - X_9^{37451} \\
& - X_9^{37454} + X_9^{37456} - X_9^{37485} + X_9^{37491} + X_9^{37523} + X_9^{37526} - X_9^{37528} - X_9^{37666} + X_9^{37677} \\
& + X_9^{37738} - X_9^{37749} + X_9^{37773} - X_9^{37779} - X_9^{37845} + X_9^{37851},
\end{aligned}
$$

$$
\begin{aligned}
I_{12}^{33} = {}& I_9^{33} + X_9^{399} - X_9^{402} - X_9^{426} + X_9^{429} - 2X_9^{444} + X_9^{454} - X_9^{457} + X_9^{478} - X_9^{508} + X_9^{534} \\
& - 2X_9^{538} + X_9^{539} + X_9^{564} - X_9^{597} - X_9^{694} - X_9^{727} + X_9^{729} - X_9^{768} + X_9^{806} + X_9^{833} - X_9^{835} \\
& - X_9^{845} + X_9^{885} - X_9^{926} + X_9^{955} + X_9^{968} - X_9^{1050} + X_9^{1095} - X_9^{1141} + X_9^{1188} - X_9^{1231} + X_9^{1235} \\
& - X_9^{1236} - X_9^{1284} - 2X_9^{1329} + X_9^{1334} + 2X_9^{1380} - X_9^{1385} + X_9^{1558} - X_9^{1560} + X_9^{1617} - X_9^{1673} \\
& - 2X_9^{1720} + X_9^{1730} - X_9^{1788} + X_9^{1847} - X_9^{1907} + X_9^{2086} - 2X_9^{2090} + 2X_9^{2154} + X_9^{2214} - 2X_9^{2219} \\
& - X_9^{2280} + 2X_9^{2285} - X_9^{2486} - X_9^{2781} + X_9^{2855} - X_9^{2930} - 3X_9^{36061} + 2X_9^{36070} + X_9^{36072} \\
& + X_9^{36091} + 2X_9^{36096} - 3X_9^{36105} - X_9^{36120} + 3X_9^{36128} - 4X_9^{36130} + X_9^{36137} + 2X_9^{36139} \\
& + 3X_9^{36167} - X_9^{36174} - 2X_9^{36176} - X_9^{36197} - X_9^{36200} - 2X_9^{36202} + X_9^{36209} + 2X_9^{36211} \\
& + X_9^{36226} + 3X_9^{36237} - 3X_9^{36243} + X_9^{36266} - X_9^{36295} + X_9^{36364} - X_9^{36399} - X_9^{36421} + X_9^{36430} \\
& - 2X_9^{36469} + X_9^{36502} + X_9^{36504} + X_9^{36527} - X_9^{36534} - X_9^{36536} + 3X_9^{36565} - X_9^{36574} - X_9^{36595} \\
& - 2X_9^{36600} + X_9^{36609} + X_9^{36624} - 2X_9^{36632} + 3X_9^{36634} - X_9^{36643} + X_9^{36683} - X_9^{36716} + X_9^{36748} \\
& - 3X_9^{36779} + X_9^{36786} + X_9^{36788} + X_9^{36809} + 2X_9^{36814} - X_9^{36821} - X_9^{36823} - X_9^{36838} - 3X_9^{36849} \\
& + 2X_9^{36855} - X_9^{36876} + 2X_9^{36884} - X_9^{36886} + X_9^{36911} - 2X_9^{36914} + X_9^{36916} + 3X_9^{36943} \\
& - 2X_9^{36945} - X_9^{37011} + X_9^{37046} + X_9^{37068} - X_9^{37077} - X_9^{37105} + 2X_9^{37114} + X_9^{37116} + X_9^{37140} \\
& - 3X_9^{37149} + X_9^{37172} - 3X_9^{37174} + X_9^{37181} + 2X_9^{37183} - X_9^{37207} - X_9^{37212} + 2X_9^{37242} \\
& - X_9^{37271} + X_9^{37274} - X_9^{37276} + X_9^{37319} - X_9^{37326} - 2X_9^{37328} - X_9^{37354} + X_9^{37361} + 2X_9^{37363} \\
& + 2X_9^{37389} - 3X_9^{37395} + X_9^{37416} + X_9^{37421} - X_9^{37424} + 2X_9^{37426} - X_9^{37451} + X_9^{37454} \\
& - 3X_9^{37456} - 2X_9^{37483} + 3X_9^{37485} + X_9^{37491} - X_9^{37526} + X_9^{37561} - X_9^{37590} + X_9^{37648} - X_9^{37683} \\
& + X_9^{37718} - X_9^{37738} - X_9^{37749} + X_9^{37773} + X_9^{37779} - 2X_9^{37808} + X_9^{37843} - X_9^{37878} + X_9^{37913},
\end{aligned}
$$

$$
\begin{aligned}
I_{12}^{34} = {}& I_9^{34} + X_9^{399} - 3X_9^{402} + X_9^{404} + 2X_9^{424} - 2X_9^{426} + 2X_9^{429} - X_9^{444} + X_9^{447} + X_9^{454} \\
& - 3X_9^{457} + 2X_9^{478} + X_9^{480} - X_9^{490} - X_9^{505} - 3X_9^{508} + X_9^{515} + X_9^{518} - 2X_9^{538} + 2X_9^{539} \\
& + X_9^{564} + 2X_9^{569} - X_9^{577} - 2X_9^{597} - 2X_9^{600} - X_9^{6694} - X_9^{6736} + 2X_9^{6741} - X_9^{6781} + X_9^{6786}
\end{aligned}
$$

$$
\begin{aligned}
& - 2X_9^{6788} + 2X_9^{6871} - X_9^{6884} - 2X_9^{6966} + X_9^{6974} + X_9^{7009} - X_9^{7011} + 2X_9^{7013} + 2X_9^{7072} \\
& - X_9^{7077} - X_9^{7119} - X_9^{7158} + X_9^{7162} - X_9^{7164} + 3X_9^{7166} - 2X_9^{7207} + X_9^{7220} + 2X_9^{7254} \\
& - X_9^{7262} - X_9^{7297} + X_9^{7299} - 2X_9^{7301} - 2X_9^{7453} + X_9^{7458} - X_9^{7498} + 2X_9^{7500} - X_9^{7502} \\
& + X_9^{7539} - 2X_9^{7547} + 2X_9^{7588} - X_9^{7601} + X_9^{7633} - 3X_9^{7635} + X_9^{7637} + X_9^{7643} + X_9^{7682} \\
& - X_9^{23713} + 2X_9^{23839} - X_9^{23856} + X_9^{23911} - 3X_9^{23947} + 2X_9^{23964} - X_9^{23999} + X_9^{24235} \\
& - X_9^{24252} + 2X_9^{24287} - X_9^{24395} - X_9^{24487} - 2X_9^{24523} + 3X_9^{24540} - X_9^{24575} + 2X_9^{24613} \\
& - X_9^{24630} - X_9^{24666} - X_9^{24739} - X_9^{24773} + X_9^{24792} - X_9^{24991} + 2X_9^{25008} - X_9^{25134} + 2X_9^{25189} \\
& - 3X_9^{25206} + X_9^{25242} + X_9^{25351} + X_9^{25387} - 2X_9^{25404} - 2X_9^{25439} - 3X_9^{25477} + 2X_9^{25494} \\
& + X_9^{25530} + X_9^{25565} + X_9^{25603} + 3X_9^{25637} - X_9^{25656} - X_9^{25673} + X_9^{25765} - X_9^{25782} + 2X_9^{26267} \\
& - 2X_9^{26357} - X_9^{26393} + X_9^{26519} + X_9^{26645} - X_9^{39248} + X_9^{39254} + X_9^{39271} + X_9^{39275} - 2X_9^{39277} \\
& + X_9^{39284} + X_9^{39320} - X_9^{39326} - X_9^{39343} - X_9^{39347} + 2X_9^{39349} - X_9^{39356} - X_9^{39384} + 2X_9^{39390} \\
& + X_9^{39399} - 3X_9^{39405} + 2X_9^{39412} + X_9^{39426} - X_9^{39433} + X_9^{39472} - X_9^{39487} - 2X_9^{39491} + X_9^{39493} \\
& + X_9^{39514} - X_9^{39521} - X_9^{39544} - X_9^{39550} + X_9^{39559} + 3X_9^{39563} - X_9^{39572} - 2X_9^{39586} + 2X_9^{39593} \\
& + X_9^{39616} - X_9^{39623} - 2X_9^{39627} + X_9^{39629} + X_9^{39642} - X_9^{39649} + X_9^{39702} - X_9^{39715} - X_9^{39717} \\
& + X_9^{39724} + X_9^{39738} - X_9^{39745} - X_9^{39774} + X_9^{39779} + X_9^{39781} - X_9^{39788} - X_9^{39794} + X_9^{39801},
\end{aligned}
$$

$$
\begin{aligned}
I_{12}^{35} = {} & I_9^{35} - 3X_9^{399} + X_9^{402} + X_9^{404} + 2X_9^{424} + 2X_9^{426} - 4X_9^{444} + 2X_9^{447} - 2X_9^{454} - 2X_9^{457} \\
& - 2X_9^{478} + X_9^{480} - X_9^{490} - 2X_9^{505} - 2X_9^{508} + 2X_9^{518} + 4X_9^{534} + 2X_9^{538} - 3X_9^{564} + 4X_9^{569} \\
& - 2X_9^{577} - 2X_9^{597} - 4X_9^{600} - 5X_9^{6694} + X_9^{6717} + 5X_9^{6741} - X_9^{6761} - 12X_9^{6775} + 5X_9^{6781} \\
& - 3X_9^{6786} + 2X_9^{6799} - X_9^{6808} + 2X_9^{6835} - X_9^{6860} + 13X_9^{6871} - 8X_9^{6877} + 4X_9^{6882} - 4X_9^{6884} \\
& - X_9^{6895} + 2X_9^{6904} + 2X_9^{6916} - X_9^{6942} - 2X_9^{6966} + 6X_9^{6970} - 4X_9^{6972} + 5X_9^{6974} - X_9^{6997} \\
& - 5X_9^{7009} + 3X_9^{7011} - 2X_9^{7013} + X_9^{7035} + 5X_9^{7072} + X_9^{7088} - X_9^{7097} - 5X_9^{7119} + X_9^{7142} \\
& + 4X_9^{7158} + X_9^{7162} - 3X_9^{7164} + 6X_9^{7166} - 2X_9^{7180} + X_9^{7189} - 12X_9^{7207} + 4X_9^{7213} - 6X_9^{7218} \\
& + 7X_9^{7220} - X_9^{7231} + X_9^{7240} + X_9^{7254} - 4X_9^{7258} + 8X_9^{7260} - 8X_9^{7262} + X_9^{7276} - 2X_9^{7285} \\
& + 3X_9^{7297} - 5X_9^{7299} + 2X_9^{7301} + X_9^{7323} - 2X_9^{7353} + X_9^{7379} + 2X_9^{7392} - X_9^{7417} - 2X_9^{7447} \\
& - 6X_9^{7453} + 4X_9^{7458} - 3X_9^{7460} - X_9^{7471} + X_9^{7480} + 2X_9^{7500} + X_9^{7502} - X_9^{7525} + 2X_9^{7537} \\
& - 5X_9^{7547} + 2X_9^{7563} - X_9^{7572} + 11X_9^{7588} - 3X_9^{7601} + X_9^{7614} - X_9^{7623} - 2X_9^{7633} - 2X_9^{7635} \\
& - X_9^{7637} + 4X_9^{7643} - X_9^{7659} + 2X_9^{7668} + X_9^{7682} - X_9^{7706} - X_9^{7762} + X_9^{7800} - 5X_9^{23713} \\
& + 5X_9^{23839} - 2X_9^{23856} - X_9^{23911} - 3X_9^{23947} + X_9^{23964} + X_9^{23999} + 4X_9^{24109} + 4X_9^{24199} \\
& - 3X_9^{24235} + X_9^{24252} + X_9^{24287} - 6X_9^{24361} - 2X_9^{24395} - 9X_9^{24487} + X_9^{24523} + 2X_9^{24540} \\
& + 3X_9^{24575} + 5X_9^{24613} - 4X_9^{24630} - X_9^{24666} - 3X_9^{24701} + X_9^{24739} + 3X_9^{24773} + 8X_9^{24792} \\
& - 4X_9^{24809} + 5X_9^{25008} - 3X_9^{25134} + 4X_9^{25189} - 6X_9^{25206} + 5X_9^{25242} - 2X_9^{25277} + 5X_9^{25351} \\
& - 4X_9^{25387} - 5X_9^{25404} - 5X_9^{25439} - 6X_9^{25477} + 3X_9^{25494} + X_9^{25530} + 5X_9^{25565} + 2X_9^{25603} \\
& + 4X_9^{25637} - 5X_9^{25656} - X_9^{25673} + 3X_9^{25765} - X_9^{25782} + 4X_9^{25818} - X_9^{25853} + 3X_9^{26233} \\
& + 5X_9^{26267} - X_9^{26357} + 2X_9^{26376} - 5X_9^{26393} + 3X_9^{26519} - X_9^{26611} - X_9^{26645} - 3X_9^{26664} \\
& - X_9^{27313} + X_9^{27456} + X_9^{27565} - 2X_9^{27655} + X_9^{27691} - X_9^{27708} - X_9^{27743} + 2X_9^{27798} - X_9^{27834} \\
& - X_9^{28105} + X_9^{28139} + X_9^{28195} - 2X_9^{28229} + X_9^{28248} + X_9^{28265} - X_9^{28338} + X_9^{28735} - X_9^{28878} \\
& + X_9^{28951} - 2X_9^{28987} + X_9^{29077} - X_9^{29094} + 2X_9^{29130} + X_9^{29165} - X_9^{29220} + X_9^{29401} - X_9^{29491} \\
& + X_9^{29525} - X_9^{29544} - 2X_9^{29561} + X_9^{29634} + X_9^{29651} + X_9^{30119} - X_9^{30209} + 4X_9^{36070} - 4X_9^{36072} \\
& - 2X_9^{36091} + 4X_9^{36096} - 4X_9^{36120} + 6X_9^{36125} - 8X_9^{36128} - 2X_9^{36137} + 2X_9^{36139} - 4X_9^{36167} \\
& - 4X_9^{36174} + 4X_9^{36176} + 6X_9^{36192} - 4X_9^{36197} + 10X_9^{36200} - 4X_9^{36202} + 2X_9^{36209} - 2X_9^{36211} \\
& - 2X_9^{36226} - 5X_9^{36263} - 8X_9^{36266} + 8X_9^{36268} + 2X_9^{36297} - 2X_9^{39248} + X_9^{39271} + 2X_9^{39275} \\
& - 4X_9^{39277} + 3X_9^{39284} + X_9^{39320} - X_9^{39343} - 3X_9^{39347} + 6X_9^{39349} - 5X_9^{39356} - X_9^{39384} \\
& + 2X_9^{39390} + X_9^{39399} - 6X_9^{39405} + 8X_9^{39412} + X_9^{39426} - X_9^{39433} + X_9^{39472} - 6X_9^{39491} \\
& + 6X_9^{39493} + 2X_9^{39514} - 2X_9^{39521} - X_9^{39544} - X_9^{39550} + X_9^{39559} + 10X_9^{39563} - 8X_9^{39565}
\end{aligned}
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\begin{aligned}
&- 5X_9^{39572} - 4X_9^{39586} + 4X_9^{39593} + X_9^{39616} - X_9^{39623} - 8X_9^{39627} + 7X_9^{39629} + 4X_9^{39642} \\
&- X_9^{39649} + X_9^{39702} - 3X_9^{39715} + 6X_9^{39724} + 2X_9^{39738} - 2X_9^{39745} - X_9^{39774} + 5X_9^{39779} \\
&- X_9^{39781} - 5X_9^{39788} - 3X_9^{39794} + X_9^{39896} - 2X_9^{39936} - X_9^{39942} + 2X_9^{39968} + 2X_9^{39987} \\
&- X_9^{39989} + X_9^{40024} + 2X_9^{40030} - 2X_9^{40056} - 2X_9^{40062} - 4X_9^{40075} + 2X_9^{40077} + X_9^{40084} \\
&+ X_9^{40096} + 4X_9^{40107} - 2X_9^{40109} - X_9^{40129} - X_9^{40166} + 2X_9^{40198} + 2X_9^{40211} - X_9^{40213} \\
&- 2X_9^{40220} - X_9^{40238} - 4X_9^{40243} + 2X_9^{40245} + 2X_9^{40252} + 2X_9^{40265} + 2X_9^{40283} - X_9^{40285} \\
&- 2X_9^{40297} - X_9^{40375} + 2X_9^{40415} - X_9^{40419} + 2X_9^{40421} - 2X_9^{40447} - X_9^{40466} - X_9^{40503} \\
&+ 2X_9^{40507} - 4X_9^{40509} - X_9^{40516} + 2X_9^{40535} - 2X_9^{40539} + 4X_9^{40541} + 2X_9^{40554} + X_9^{40561} \\
&- X_9^{40575} - 2X_9^{40586} - X_9^{40643} + 2X_9^{40645} + 2X_9^{40652} + 2X_9^{40675} - 4X_9^{40677} - 2X_9^{40684} \\
&- X_9^{40690} - 2X_9^{40697} - X_9^{40715} + 2X_9^{40717} + 2X_9^{40722} + 2X_9^{40729} - X_9^{40762} - X_9^{40836} \\
&+ 2X_9^{40868} + X_9^{40881} - X_9^{40908} - 2X_9^{40913} + X_9^{40953}.
\end{aligned}
$$